Theory and Design of Soundfield Reproduction using Continuous Loudspeaker Concept

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Abstract

Reproduction of a soundfield is a fundamental problem in acoustic signal processing. A common approach is to use an array of loudspeakers to reproduce the desired field where the least squares method is used to calculate the loudspeaker weights. However, the least squares method involves matrix inversion which may lead to errors if the matrix is poorly conditioned. In this paper, we use the concept of theoretical continuous loudspeaker on a circle to derive the discrete loudspeaker aperture functions by avoiding matrix inversion. In addition, the aperture function obtained through continuous loudspeaker method reveals the underlying structure of the solution as a function of the desired soundfield, the loudspeaker positions, and the frequency. This concept can also be applied for the 3D soundfield reproduction using spherical harmonics analysis with a spherical array. Results are verified through computer simulations.

Index Terms

soundfield reproduction, least squares method, matrix inversion, continuous loudspeaker, array of loudspeakers

I. Introduction

The ability to control the soundfield within a given region of space is a fundamental problem in acoustic signal processing. It is possible for listeners to spatialize sound and being immersed in a realistic, yet

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virtual sound environment using an array of loudspeakers [1]. Early studies of soundfield reproduction were performed by Gerzon [2], in which he produced the first-order cylindrical harmonics in terms of a plane wave soundfield at a single point in space. It was then developed as Higher Order Ambisonics (HOA) [3], [4] for larger reproduction areas [5]. The principle of Ambisonics system is based on the expansion of the desired soundfield and the soundfields of the secondary sources into spatial harmonics [6], [7]. This is called as "mode-matching" approach [6].

Wave field synthesis (WFS) is another technique used for soundfield reproduction. It was firstly introduced by Berkhout [8], [9], who used the holographic technique, known as wavefield synthesis, to reproduce a desired soundfield over a relatively large area using a relatively large number of loudspeakers. The WFS approaches [10], [11], [12], [13], [14] are based on the Kirchhoff-Helmholtz integral [7] [15]. The fundamental principle of WFS states that a continuous distribution of appropriately driven secondary monopole and dipole sources arranged on the boundary of the desired listening area is capable to reproduce any virtual wave field [7]. For practical reasons, the continuous distribution of secondary sources is replaced by an arrangement of loudspeakers at discrete positions by spatial sampling process [16]. To avoid spatial aliasing, typical WFS systems employ loudspeakers with a spacing of 10cm to 30cm [15], this approximately corresponds to half a wavelength separation between the loudspeakers [9], [12], [10]. In order to fulfill this interval requirement, a large number of loudspeakers are placed on the boundary of the desired listening area. However, in this paper we consider the reproduction of localized soundfield using a number of loudspeakers without the constraints to place them on the boundary of the desired region. The connection between the WFS and the Ambisonics approach is investigated in [17].

The least squares is another popular approach to derive loudspeaker weights for soundfield reproduction. It is largely used in active control applications [14], [18]. The least squares approach was first considered in 1964 by Trott [19] who created a single-frequency plane wave in the near field of a "dense" planar hydrophone array to calibrate an unknown transducer. In 90's, Kirkeby [20], [21] proposed the least squares techniques to determine the theoretical minimum number of loudspeakers required to produce a soundfield locally. Most recent works in soundfield reproduction such as [1], [22], [23], [24], [25] are based on the least squares method due to the advantage that the sound is matched over a region of space rather than at a single point. However, the least squares approach involves a matrix inversion, if the matrix is poorly conditioned, the solution may not exist. The condition number of the matrix dictates the sensitivity of errors of the resulting solution [26], [27].

To avoid the matrix-inversion, we apply the continuous sensor concept used in beamforming of the sensor arrays [28] [29] and develop the theoretical *continuous loudspeaker* [30] [31] in soundfield reproduction, which has an aperture that is a function of space variables and frequency. An exact relationship between the desired soundfield and the aperture function is derived. From this vintage point, we see that a discrete array with a finite number of loudspeakers, which exhibits an approximate aperture, is readily derived from the continuous loudspeaker theory. We also show that this approximation is exact if the desired soundfield is "mode limited". By using this continuous loudspeaker concept, we obtain the loudspeaker aperture function without using matrix inversion. In addition, the aperture function obtained through the continuous loudspeaker method reveals the underlying structure of the solution as a function of the desired soundfield, the loudspeaker positions and the frequency. However, the least squares inversion provides the numerical information but hides these physical insights of the solution. This approach can be considered as a combination of the continuous concept from Wave Field Synthesis (WFS) and the mode-matching concept from the Ambisonics system (Spherical Harmonics Analysis).

In this paper, we focus on finding the proper loudspeaker weights to reproduce 2D soundfield in free space (i.e. we ignore the effect of reverberation). Meanwhile the performance bounds on the 2D soundfield reproduction [1] is verified by this continuous loudspeaker method. Specifically, the relationships between the number of loudspeakers, the size of the reproduction region, the frequency range, and the desired accuracy are in correspondence with the results obtained in [1]. Simulation results demonstrate the favorable performance for both planewave source and cylindrical wave source soundfield reproduction. This continuous loudspeaker concept can be also applied in the 3D soundfield reproduction by spherical harmonics analysis using a spherical array of loudspeakers. In addition, we show that only a finite number of loudspeakers are necessary to implement the continuous loudspeaker to reproduce a mode limited soundfield.

The paper is organized as follows. In section II, we present the system model for soundfield reproduction using a continuous loudspeaker. The loudspeaker weight design is described in Section III. The theoretical continuous aperture is firstly introduced, followed by the approximation to a practical discrete array of loudspeakers by spatial sampling. Least squares method is presented in Section IV for comparison. In Section V, we apply the continuous loudspeaker concept into the 3D soundfield reproduction. Finally, we present the simulations result to compare the performance of the continuous loudspeaker methods and the least squares methods.

Notation

Throughout this paper, we use the following notations: matrices and vectors are represented by upper and lower bold face respectively, e.g., $\boldsymbol{X}$ and $\boldsymbol{x}$. A unit vector in the direction $\boldsymbol{x}$ is denoted by $\hat{\boldsymbol{x}}$, i.e., $\hat{\boldsymbol{x}} = \boldsymbol{x}/\|\boldsymbol{x}\|$, where $\|\cdot\|$ denotes the Euclidean distance ($\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$) and "$\cdot$" is used to represent the inner product of two vectors. The vector length is denoted by x, i.e., $\boldsymbol{x} = x\hat{\boldsymbol{x}}$ or $x = \|\boldsymbol{x}\|$. The symbol $i = \sqrt{-1}$ is used to denote the imaginary part of complex numbers.

II. System Model

In this paper, we mainly concentrate on 2D or height invariant soundfields except in Section V where we show how to extend the result to 3D soundfields. Let r be the radius of a 2D circular spatial reproduction region, whose origin is at O as shown in Figure 1. Any arbitrary observation point within this circular spatial reproduction region is denoted as $\boldsymbol{x} \equiv (\|\boldsymbol{x}\|, \phi_x)$. We assume that the loudspeakers are placed on a circle with radius $R > r$. The weight of a particular loudspeaker located at angle ϕ is denoted as $\rho(\phi, k)$, where $k = 2\pi f/c$ is the wavenumber, f is the frequency, and c is the speed of sound propagation. In general case, we assume c is 340 m/s in our simulations[1]. Throughout this paper, we use k instead of f to represent frequency since we assume constant c. The lower limit and upper limit of the desired frequency band is denoted by k_ℓ and k_u respectively.

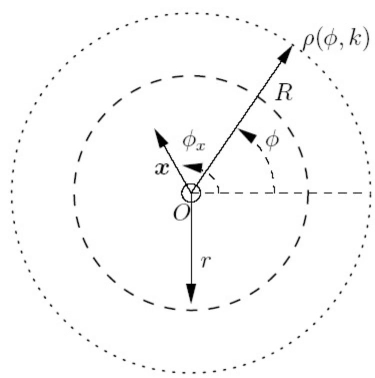

Fig. 1: System model used in this paper. Any arbitrary observation point within the desired circular spatial reproduction region of radius r is denoted as $\boldsymbol{x} \equiv (\|\boldsymbol{x}\|, \phi_x)$. The loudspeakers are placed on a circle with radius $R > r$. The weight of a particular loudspeaker located at angle ϕ is denoted as $\rho(\phi, k)$, where k is the wavenumber.

[1]The actual speed of sound is a function of many parameters, e.g. temperature.

A. Truncation Theorem

We can represent an arbitrary 2D (height invariant) soundfield $S(\boldsymbol{x}, k)$ generated by any number of sound sources outside of a region of radius r, having the following representation[2]:

$$S(\boldsymbol{x}, k) = \sum_{m=-\infty}^{\infty} \alpha_m(k) J_m(k\|\boldsymbol{x}\|) e^{im\phi_x}, \tag{1}$$

where $J_m(\cdot)$ is the Bessel function of order m, and $\alpha_m(k)$ are a set of coefficients for the soundfield. Note, the representation (1) is in the form of a Fourier series expansion. Any arbitrary 2D (height invariant) soundfield due to any number of cylindrical waves and/or planewaves can be represented by (1).

The general representation of an arbitrary soundfield (1) can be constructed by solving the Helmholtz wave equation in cylindrical coordinate system [32, p.67]. We can see that any arbitrary soundfiled is a weighted sum of basis functions $J_m(k\|\boldsymbol{x}\|)e^{im\phi_x}$, where m is referred to as the mode.

The representation (1) has an infinite number of orthogonal modes, however, we can truncate this series expansion to a finite number within the region of interest due to the properties of the Bessel functions and the fact that the soundfield has to be bounded within a spatial region where all sources are outside [33]. Hence, (1) can be truncated to $|m| \leq M$ terms as:

$$S_M(\boldsymbol{x}, k) = \sum_{m=-M}^{M} \alpha_m(k) J_m(k\|\boldsymbol{x}\|) e^{im\phi_x}, \tag{2}$$

where the normalized truncation error is upper bounded by

$$\epsilon_M(\boldsymbol{x}) = \frac{|S(\boldsymbol{x}, k) - S_M(\boldsymbol{x}, k)|}{2\pi |S(\mathbf{0}, k)|} \leq \eta e^{-\delta}, \tag{3}$$

provided that the truncation length is chosen as

$$M = \lceil e\|\boldsymbol{x}\|k/2 \rceil, \tag{4}$$

where $\eta \approx 0.16127$ and δ is a positive integer [33].

The truncation theorem states that the relative truncation error is no more than 16.1% once M equals to the critical threshold[3] $\lceil e\|\boldsymbol{x}\|k/2\rceil$, and thereafter decreases at least exponentially to zero as M increases. In other words, a larger truncation depth only marginally increases the accuracy. We refer readers to [33] for the proof.

[2]We assume no sound sources or scattering objects being present inside the reproduction area.

[3]This is a mathematical measure, which has not been connected to any perceptual studies yet in the literature.

B. Desired Soundfield

We apply the truncation theorem for the desired soundfield within a circular region of radius r. The desired soundfield at a point $\boldsymbol{x} \equiv (\|\boldsymbol{x}\|, \phi_x)$ from O now becomes

$$S^{\mathrm{d}}(\boldsymbol{x}, k) = \sum_{m=-M}^{M} \alpha_m^{(\mathrm{d})}(k) J_m(k\|\boldsymbol{x}\|) e^{im\phi_x}, \tag{5}$$

where $M = \lceil erk/2 \rceil$ and $\alpha_m^{(\mathrm{d})}(k)$ uniquely represents the desired field. Note that the truncation number M is dependent on frequency k. For a fixed region of radius r, the higher the frequency, the higher the truncation number M is. Thus, if we choose a truncation number corresponding to the highest frequency k_u and the radius of the desired region, this truncation number should be sufficient to have less errors for all other frequencies and all the points within the region. Hence, the problem we consider in this paper is as follows: *Given a soundfield by a finite number of components $\alpha_m^{(d)}(k)$ for $m = -M, \ldots, M$, and k $\in$ [k_ℓ,k_u], where $M = \lceil erk/2 \rceil$, how can we reconstruct the field in a spatial region of radius r?*

We illustrate simple desired soundfields below as examples:

1) Desired Soundfield for a planewave source: For a planewave originating from direction $\hat{\boldsymbol{\phi}}_{\mathrm{pw}}$, the desired soundfield is[4]

$$S^{\mathrm{d}}(\boldsymbol{x}, k) = e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{\phi}}_{\mathrm{pw}}}. \tag{6}$$

Following the convention in [34] of assuming a time dependency $e^{-i\omega t}$, we use the Jacobi-Anger expression [32, p.67] to write,

$$e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{\phi}}_{\mathrm{pw}}} = \sum_{m=-\infty}^{\infty} i^m e^{-im\phi_{\mathrm{pw}}} J_m(k\|\boldsymbol{x}\|) e^{im\phi_x}, \tag{7}$$

where ϕ_{pw} is the polar angle of $\hat{\boldsymbol{\phi}}_{\mathrm{pw}}$, i.e., $\hat{\boldsymbol{\phi}}_{\mathrm{pw}} \equiv (1, \phi_{\mathrm{pw}})$ and $\boldsymbol{x} = (\|x\|, \phi_x)$.

Thus, the desired soundfield coefficients for a planewave source, $\alpha_m^{(\mathrm{d})}(k)$ are given by

$$\alpha_m^{(\mathrm{d})}(k) = i^m e^{-im\phi_{\mathrm{pw}}}. \tag{8}$$

Typically only a finite number of these cofficients, i.e., $\{\alpha_m^{(\mathrm{d})}(k)\}_{-M}^{M}$ are chosen according to the truncation theorem where $M = \lceil erk/2 \rceil$ with r being the radius of the desired reproduction region.

[4]We assume the wave vector of the plane wave has no component perpendicular to the plane on which the loudspeaker are arranged.

2) Desired Soundfield for a cylindrical wave source: For a cylindrical wave source[5] of unit strength located at $\boldsymbol{y}$, with $\|\boldsymbol{y}\| > \|\boldsymbol{x}\|$, the desired soundfield within the circular zone at a point $\boldsymbol{x}$ from O due to the sound source at $\boldsymbol{y}$ is given by [32, p.66], [23]:

$$S^{\mathrm{d}}(\boldsymbol{x}, k) = \frac{i}{4} H_0^{(1)}(k\|\boldsymbol{y} - \boldsymbol{x}\|), \tag{9}$$

where $H_0^{(1)}(k\| \cdot \|)$ is the zeroth order Hankel functions of the first kind[6]. Note that the fundamental solution to the Helmholtz wave equation [32, p.66] in 2D is $(i/4)H_0^{(1)}(k\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|)$, where the source is at $R\hat{\boldsymbol{\phi}}$ and the observation point is at $\boldsymbol{x}$, and $H_m^{(1)}(\cdot)$ is the Hankel function of the first kind[7].

We use the addition property of the Hankel function [32, p.67] to write

$$H_0^{(1)}(k\|\boldsymbol{y} - \boldsymbol{x}\|) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(k\|\boldsymbol{y}\|)e^{-im\phi_y} J_m(k\|\boldsymbol{x}\|)e^{im\phi_x} \text{ for } \|\boldsymbol{y}\| > \|\boldsymbol{x}\|, \tag{10}$$

where ϕ_y is the polar angle of the monopole source at point $\boldsymbol{y}$.

Thus, the desired soundfield coefficients for a monopole source, $\alpha_m^{(\mathrm{d})}(k)$ are given by

$$\alpha_m^{(\mathrm{d})}(k) = H_m^{(1)}(k\|\boldsymbol{y}\|)e^{-im\phi_y}, \tag{11}$$

with an appropriate number of coefficients $m = -M, \ldots, M$, depending on the size of the reproduction region.

Any desired soundfield can be constructed from a weighted combination of (8) or (11).

C. Actual Soundfield

Assume that no loudspeaker is placed inside the spatial zone of interest. Let $\rho(\phi, k)$ be the aperture function of the circular continuous loudspeaker with radius R to achieve the desired field. Thus, the reproduced soundfield within the circular zone at a point $\boldsymbol{x}$ from O is given by

$$S^{\mathrm{a}}(\boldsymbol{x}, k) = \int_0^{2\pi} \rho(\phi, k) \frac{i}{4} H_0^{(1)}(k\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|) d\phi, \tag{12}$$

where $R\hat{\boldsymbol{\phi}} \equiv (R, \phi)$ and it is used to represent the loudspeaker position.

[5]We assume an infinite long point cylindrical wave (line) source which is perpendicular to the horizontal plane. In 2D point of view, this could be treated as a point source.

[6]For a 2D line source, or a cylindrical source, $\frac{i}{4}H_0^{(1)}(kr)$ gives the field at a distance r from the source [32]; for a 3D line source, this is equal to the spherical Hankel functions of the first kind $h_0^{(1)}(kr) = e^{ikr}/kr$.

[7]This is different from spherical wave fronts from a point source in 3D, where the fundamental solution to the wave equation in 3D is $\exp(ik\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|)/4\pi\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|$.

For simplicity, here we have assumed that the loudspeakers are infinitely long point cylinders perpendicular to the 2D plane of interest[8].

Provided $R > \|\boldsymbol{x}\|$, similar to (10), we use the addition theorem for cylindrical harmonics [34, p.267] to write

$$H_0^{(1)}(k\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(kR)e^{-im\phi}J_m(k\|\boldsymbol{x}\|)e^{im\phi_x}. \tag{13}$$

Since the aperture function $\rho(\phi, k)$ is a periodic function of ϕ, we use the Fourier series expansion to write

$$\rho(\phi, k) = \sum_{m'=-\infty}^{\infty} \beta_{m'}(k)e^{im'\phi}, \tag{14}$$

where $\beta_m(k)$ are the Fourier coefficients.

By substituting (13) and (14) into (12), and evaluating the integral, we get

$$S^{\mathrm{a}}(\boldsymbol{x}, k) = \sum_{m=-\infty}^{\infty} \beta_m(k)\frac{i}{2}\pi H_m^{(1)}(kR)J_m(k\|\boldsymbol{x}\|)e^{im\phi_x}. \tag{15}$$

Note that $S^{\mathrm{a}}(\boldsymbol{x}, k)$ is a weighted sum of basis functions $J_m(k\|\boldsymbol{x}\|)e^{im\phi_x}$, where the m-mode weight is $\beta_m(k)\frac{i}{2}\pi H_m^{(1)}(kR)$.

III. Loudspeaker Weight Design

We begin this section by deriving the continuous loudspeaker aperture, in order to develop an exact relationship between the desired soundfield and the aperture function. Then, we address the engineering problem of physically reproducing the soundfield using an array of loudspeakers. The aperture function of the continuous loudspeaker will be approximated by a discrete loudspeaker array to realize practical implementation.

A. Theoretical Continuous Aperture

To design the loudspeaker aperture function, we equate the desired soundfield (5) to the actual soundfield (15), i.e.,

$$S^{\mathrm{d}}(\boldsymbol{x}, k) = S^{\mathrm{a}}(\boldsymbol{x}, k), \tag{16}$$

for $\boldsymbol{x} \in$ desired region and $k \in [k_\ell, k_u]$.

Using (5) and (15), we have for $m = -M, \ldots, M$,

[8]In practice, loudspeakers are not infinitely long point cylinders, suitable near-field correction should be used. The readers may refer to [31], [35] and [15] for an in-depth treatment of similar finite length/near-field correction.

$$\beta_m(k) = \frac{2}{i\pi H_m^{(1)}(kR)} \alpha_m^{(\mathrm{d})}(k), \tag{17}$$

where $\beta_m(k)$ are the Fourier series coefficients of the aperture function $\rho(\phi, k)$. We state this result as a theorem for soundfield reproduction using a theoretical continuous circular loudspeaker:

Theorem 1: If a desired soundfield in a 2D spatial region is given by the cylindrical harmonics of the soundfield $\alpha_m^{(\mathrm{d})}(k)$, for $m = -M, \ldots, M$, where $M = \lceil ker/2 \rceil$, and frequency band $k \in [k_\ell, k_u]$, then the aperture function of the theoretical continuous circular loudspeaker of radius R, which reconstructs the soundfield exactly, is given by:

$$\rho(\phi, k) = \sum_{m=-M}^{M} \frac{2}{i\pi H_m^{(1)}(kR)} \alpha_m^{(\mathrm{d})}(k) e^{im\phi}. \tag{18}$$

Note that the summation in (18) has only $2M + 1$ terms since the desired soundfiled is mode limited to M. That is, only the lower order $2M + 1$ cylindrical harmonic coefficients are used to describe the soundfield. It assumes that the desired soundfield has zero energy in the higher order modes within the reproduction region.

B. Spatial Sampling of Continuous Loudspeaker

The discretization of the continuous loudspeaker is modeled by sampling the loudspeaker aperture function $\rho(\phi, k)$ in (18). Note that $\rho(\phi, k)$ is a periodic function of ϕ with period 2π. Since our desired soundfield is mode limited to M, i.e., desired soundfield coefficients are $\{\alpha_m^{(\mathrm{d})}(k)\}_{-M}^{M}$, the highest (angle) frequency is $M/(2\pi)$ corresponding to $e^{jM\phi}$. According to Shannaon sampling theorem [36], we can exactly reproduce $\rho(\phi, k)$ by its samples, if the sampling rate is greater than twice the maximum frequency.

Thus, a sampling angle frequency $\zeta_s > (M/\pi)$ should be sufficient. This equals to a sampling interval of $\triangle\phi_s = (1/\zeta_s) < (\pi/M)$. Hence we need at least $2M+1$ equidistant samples on the circle to accurately reproduce $\rho(\phi, k)$ from its samples.

In a mathematical point of view, mode limited (up to M) $\rho(\phi, k)$ can be reconstructed by its samples without aliasing by sampling at least $2M + 1$ equidistant points on a circle. However, we do not directly reconstruct the aperture function. Instead, we reconstruct the desired soundfield on the reproduction region by the samples of the aperture function. Thus, we need to investigate the effects of the sampling of $\rho(\phi, k)$ on the reconstructed soundfield. This is considered in the next section.

C. Discrete Loudspeaker Array Design

By applying the sampling procedure mentioned in the previous section, we can observe that if we have $Q \geq 2M+1$ equally spaced points (loudspeakers) on the circle, then we can exactly reproduce $\rho(\phi, k)$ from its samples $\rho(\phi_q, k)$, $q = 1, \ldots, Q$ in the frequency band $k \in [k_l,\ k_u]$. This leads us to design q^{th} loudspeaker weight as

$$w_q(k) = \rho(\phi_q, k)\triangle\phi. \tag{19}$$

where $\triangle\phi = 2\pi/Q$ is the angular spacing of the loudspeakers and $\phi_q = q\triangle\phi$. With this setting, we have the loudspeaker weights as:

$$w_q(k) = \sum_{m=-M}^{M} \frac{2}{i\pi H_m^{(1)}(kR)} \alpha_m^{(\mathrm{d})}(k) e^{im\phi_q} \triangle\phi, \ \ \text{for } q = 1, \ldots, Q. \tag{20}$$

However, with this choice we can only reproduce the desired soundfield with some errors. We quantify the errors below.

Let $S^{\mathrm{a}}_{\mathrm{disc}}(\boldsymbol{x}, k)$ be the reproduced soundfield using the discrete circular loudspeaker array with weights given in (20), then we have

$$S^{\mathrm{a}}_{\mathrm{disc}}(\boldsymbol{x}, k) = \sum_{q=1}^{Q} w_q(k) \frac{i}{4} H_0^{(1)}(k\|R\hat{\boldsymbol{\phi}}_q - \boldsymbol{x}\|), \tag{21}$$

where $\hat{\boldsymbol{\phi}}_q = (1, \phi_q)$. We substitute (20) and (13) into (21) and get

$$S^{\mathrm{a}}_{\mathrm{disc}}(\boldsymbol{x}, k) = \frac{1}{Q} \sum_{m=-\infty}^{\infty} \sum_{m'=-M}^{M} \alpha_{m'}^{(\mathrm{d})}(k) \frac{H_m^{(1)}(kR)}{H_{m'}^{(1)}(kR)} [\sum_{q=1}^{Q} e^{i(m'-m)\frac{2\pi}{Q}q}]\, J_m(k\|\boldsymbol{x}\|) e^{im\phi_x}. \tag{22}$$

From the geometric progression [37], we have

$$\sum_{q=1}^{Q} e^{i(m'-m)\frac{2\pi}{Q}q} = \begin{cases} Q & \text{if } m' = m + nQ \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

where n is an integer. By substituting (22) to (21), we obtain

$$S^{\mathrm{a}}_{\mathrm{disc}}(\boldsymbol{x}, k) = \sum_{n=-\infty}^{\infty} \sum_{m=-M-nQ}^{M-nQ} \alpha_{m+nQ}^{(\mathrm{d})}(k) \frac{H_m^{(1)}(kR)}{H_{m+nQ}^{(1)}(kR)} J_m(k\|\boldsymbol{x}\|) e^{im\phi_x}, \tag{24}$$

which is not the desired soundfield.

However, we can write (24) as

$$S^{\mathrm{a}}_{\mathrm{disc}}(\boldsymbol{x}, k) = S^{\mathrm{d}}(\boldsymbol{x}, k) + S_{\mathrm{error}}(\boldsymbol{x}, k), \tag{25}$$

where $S^{\mathrm{d}}(\boldsymbol{x},k)$ is the desired mode limited soundfield given by (5) and

$$S_{\text{error}}(\boldsymbol{x},k) = \sum_{n=-\infty,n\neq 0}^{\infty} \sum_{m=-M-nQ}^{M-nQ} \alpha^{(\mathrm{d})}_{m+nQ}(k) \frac{H^{(1)}_{m}(kR)}{H^{(1)}_{m+nQ}(kR)} J_m(k\|\boldsymbol{x}\|)e^{im\phi_x}. \tag{26}$$

We have the following remarks:

1) From (18) and (20), we notice that the underlying structure of the loudspeaker aperture function/weights is a function of desired field coefficients $\alpha^{(\mathrm{d})}_{m}(k)$, loudspeaker positions (R,ϕ_q), and frequency k. The solution based on least square methods does not reveal this underlying structure.
2) Note that the reproduction error $S_{\text{error}}(\boldsymbol{x},k)$ in (26) does not induce any errors to lower order harmonics, i.e., for $m=-M,...,M$. The errors are introduced only through higher order harmonics, i.e., $|m|>M$. This is clear from the second summation in (26) where the summation is over different sets of $2M$ higher order modes.
3) In the WFS literature, similar errors are termed as aliasing errors [7], [16]. However, in our design, there is no error involved in lower order modes due to the mode limited desired soundfield.
4) If the mode limit M is chosen according to the truncation theorem, i.e., $M=\lceil erk_u/2\rceil$ (see Section II-A), then the contribution from the higher order modes to the reproduction area is minimal since the higher order modes are relatively inactive within the desired region. Informally, this can be seen from observing (26), since $|H^{(1)}_{m}(kR)/H^{(1)}_{m+nQ}(kR)|<1$ for the allowed values of m, $\{\alpha^{(\mathrm{d})}_{m}(k)\}^{M}_{-M}$ are finite, and the high-pass nature of the higher order Bessel functions. This is true for all $k\in[k_\ell,k_u]$ as M is chosen corresponding to k_u.
5) To avoid errors within the desired region, we need $Q\geq 2\lceil erk_u/2\rceil+1$ loudspeakers at frequency k_u. Hence, the higher the upper frequency k_u, the higher the number of loudspeakers are needed. If we choose a particular Q corresponding to a particular k_u, then errors will occur in reproduction for $k>k_u$.
6) The angular spacing between adjacent loudspeakers is $\triangle\phi=2\pi/Q\leq 2\pi/(2M{+}1)=2\pi/(2\lceil erk/2\rceil+1)=2\pi/(erk)=\lambda/(er)$. Thus, the corresponding loudspeaker interval is $\triangle L=\triangle\phi\cdot R=(\lambda/e)\cdot(R/r)$. Note that this is consistent with the aliasing-free requirement of the WFS method [9], [10], [12], which is to have $\lambda/2$ separation between the loudspeakers at the edge of the reproduction region[9]. From this result, we also realize that we can move the speakers as far as

[9]We choose the truncation number $M=\lceil erk/2\rceil$ which has an upper error bound of 16.1%; however, if we choose a less restricted truncation number $M=\lceil kr\rceil$ as proposed in [1], the corresponding loudspeaker interval is approximately $\lambda/2$, which is comparable to the aliasing-free requirement of the WFS method.

we like but still need the same number $2\lceil ker/2\rceil+1$ of loudspeakers. However, in WFS method, loudspeakers are located at the boundary of the reproduction region; if we move the loudspeakers away from the reproduction region, more loudspeakers are required to achieve accurate reproduction. Nevertheless, it will result in a larger reproduction region. These two methods can be treated as a trade-off between the reproduction region and the number of loudspeakers. If a small reproduction region is required with less restriction on the location of loudspeakers, our proposed method is preferable.

IV. Alternative Least Squares Design

For comparison purposes, we outline an alternative least square design criterion in this section. The general form to reproduce the actual soundfield using discrete loudspeaker array is given by

$$S^{\mathrm{a}}(\boldsymbol{x},k)=\sum_{q=0}^{Q} w_q(k)\frac{i}{4}H_0^{(1)}(k\|R\hat{\boldsymbol{\phi}}_q-\boldsymbol{x}\|). \tag{27}$$

Using the addition theorem for cylindrical harmonics (13), we write (27) in basis expansion form as

$$S^{\mathrm{a}}(\boldsymbol{x},k)=\sum_{m=-\infty}^{\infty}\alpha_m^{(\mathrm{a})}(k)J_m(k\|\boldsymbol{x}\|)e^{im\phi_x}, \tag{28}$$

where

$$\alpha_m^{(\mathrm{a})}(k)=\sum_{q=0}^{Q} w_q(k)\frac{i}{4}H_m^{(1)}(kR)e^{-im\phi_q}. \tag{29}$$

We equate the desired soundfield to the actual soundfield to design the loudspeaker weights. I.e.,

$$S^{\mathrm{d}}(\boldsymbol{x},k)=S^{\mathrm{a}}(\boldsymbol{x},k),\ \text{for } k\in[k_\ell,k_u]. \tag{30}$$

Thus, using (5), (28) and (29), we have

$$\alpha_m^{(\mathrm{d})}(k)=\sum_{q=0}^{Q} w_q(k)\frac{i}{4}H_m^{(1)}(kR)e^{-im\phi_q}, \tag{31}$$

for $m=-\infty,\dots,\infty$, and $k\in[k_\ell,k_u]$.

To obtain the loudspeaker weights, from (31), we construct a linear system of equations:

$$\boldsymbol{\alpha}(k)=\boldsymbol{H}(k)\boldsymbol{w}(k) \tag{32}$$

where $\boldsymbol{\alpha}(k)=[\alpha_{-M}^{(\mathrm{d})}(k),\dots,\alpha_M^{(\mathrm{d})}]^T$, $\boldsymbol{w}(k)=[w_1(k),\dots,w_Q(k)]^T$, and

$$\boldsymbol{H}(k)=\frac{i}{4}\begin{pmatrix} H_{-M}^{(1)}(kR)e^{iM\phi_1} & \cdots & H_{-M}^{(1)}(kR)e^{iM\phi_Q} \\ \vdots & \ddots & \vdots \\ H_M^{(1)}(kR)e^{-iM\phi_1} & \cdots & H_M^{(1)}(kR)e^{-iM\phi_Q} \end{pmatrix}. \tag{33}$$

The loudspeaker weights can be calculated by solving the system of linear equations described by (32). The number of loudspeakers Q specifies whether the linear system (32) can be solved exactly or not. For the desired soundfield, we have $2M+1$ soundfield coefficients. If the number of loudspeaker Q matches the number of soundfield coefficients, the weight can be calculated exactly, otherwise they are estimated. There are three cases of interest [1].

a. Over-determined system (i.e., $2M+1 > Q$)

   In general no exact solution to the loudspeaker weights exists. The loudspeaker weights would typically be estimated by solving the least squares solution [38, p.236],

$$\min_{\boldsymbol{\alpha}(k)} \|\boldsymbol{H}(k)\boldsymbol{w}(k) - \boldsymbol{\alpha}(k)\|^2.$$

   Note that this least squares approach attempts to find the set of loudspeaker weights that can best reproduce, in a least squares sense, all of the spatial basis functions $m = -M, \ldots, M$. However, the lower order basis functions carry the most energy for small circular reproduction regions, with higher order modes contributing energy to larger reproduction regions.

b. Exact system (i.e.,$Q = 2M+1$)

   A unique solution to (32) exists, given by

$$\boldsymbol{w}(k) = \boldsymbol{H}(k)^{-1}\boldsymbol{\alpha}(k). \tag{34}$$

   This is achieved under the condition that $\boldsymbol{H}(k)$ is a square non-singular matrix. Although this solution will satisfy (32) exactly, it is somewhat of a moot point if $\boldsymbol{H}$ is poorly conditioned. For $\boldsymbol{H}(k)$ to be non-singular, $det[\boldsymbol{H}(k)] \neq 0$. The conditioning of $\boldsymbol{H}(k)$ is determined primarily by the loudspeaker arrangement [10].

c. Under-determined system (i.e., $Q > 2M+1$)

   In this case, there is either no solution to the loudspeaker weights or an infinite number of solutions. In the latter case the loudspeaker weight would be selected to satisfy [38, p.271],

$$\min_{\boldsymbol{w}(k)} \|\boldsymbol{w}(k)\|^2 \quad \text{subject to} \quad \boldsymbol{H}(k)\boldsymbol{w}(k) = \boldsymbol{\alpha}(k).$$

With the above mentioned restrictions, in some cases, the matrix of the least squares equation is close to singular matrix or badly scaled. The least squares inversion provides the best numerical information but hides the physical structure of the solution while the continuous loudspeaker method reveals the physical insights of the loudspeaker weights.

[10] In $3D$ case, a sparse arrangement of loudspeakers on the surface of sphere is preferable to minimize the condition number [1].

V. Application to 3D reproduction using continuous loudspeaker design

The continuous loudspeaker concept used in the Section III is appropriate for the sound reproduction of a height-invariant 2D field . We can apply this continuous loudspeaker concept into 3D sound reproduction using spherical harmonics analysis on a spherical array. Similar works have been reported in the literature by Poletti [6], Fazi [14] and Ahrens [31]. Any arbitrary soundfield at a point $\boldsymbol{x}$ and wavenumber k can be represented as [11]

$$S(\boldsymbol{x}, k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{nm}(k) j_n(k\boldsymbol{x}) Y_{nm}(\hat{\boldsymbol{x}}), \tag{35}$$

where $\alpha_{nm}(k)$ are a set of harmonic coefficients for the spherical harmonics $Y_{nm}(\hat{\boldsymbol{x}})$, $j_n(\cdot)$ is the nth order spherical Bessel function of the first kind.

Similar to the truncation theorem for the 2D soundfield reproduction, we can truncate the summation over n in the expansion (35) to a finite number of $N = \lceil e\|\boldsymbol{x}\|k/2\rceil$ using the properties of the Bessel functions [33].

With an appropriate truncation, the desired soundfield at a point $\boldsymbol{x} = x\hat{\boldsymbol{x}}$ now becomes:

$$S^{\mathrm{d}}(\boldsymbol{x}, k) = \sum_{n=0}^{N} \sum_{m=-n}^{n} \alpha^{(\mathrm{d})}_{nm}(k) j_n(k\boldsymbol{x}) Y_{nm}(\hat{\boldsymbol{x}}), \tag{36}$$

where the spherical harmonics coefficient $\alpha^{(\mathrm{d})}_{nm}(kx)$ uniquely represents the desired field.

We assume the observation point $\boldsymbol{x} = x\hat{\boldsymbol{x}}$ satisfies $x < R$, and we also assume all loudspeakers are placed on the sphere of radius R and equidistant from origin O. Let $\rho(\hat{\boldsymbol{\phi}}, k)$ be the aperture function of the spherical continuous loudspeaker with radius R to achieve the desired field[12]. The reproduced soundfield due to the continuous loudspeaker array can be written as

$$S^{\mathrm{a}}(\boldsymbol{x}, k) = \int \rho(\hat{\boldsymbol{\phi}}; k) \frac{e^{ik\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|}}{4\pi\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|} d\hat{\boldsymbol{\phi}}, \tag{37}$$

where the integration is over the unit sphere.

Provided $R > \|\boldsymbol{x}\|$, we apply the addition theorem to the 3D Helmholtz equation [32, p.30],

$$\frac{e^{ik\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|}}{4\pi\|R\hat{\boldsymbol{\phi}} - \boldsymbol{x}\|} = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(kx) h^{(1)}_n(kR) Y_{nm}(\hat{\boldsymbol{x}}) Y^{*}_{nm}(\hat{\boldsymbol{\phi}}), \tag{38}$$

where $h^{(1)}_n$ is the nth order spherical Hankel function of the first kind.

[11] Again, we assume no sound sources and scattering objects being present inside the reproduction area.

[12] The continuous loudspeaker aperture function $\rho(\hat{\boldsymbol{\phi}}; k)$ is only defined on the surface of a sphere.

We express this aperture function $\rho(\hat{\boldsymbol{\phi}};k)$ in spherical harmonics as

$$\rho(\hat{\boldsymbol{\phi}};k)=\sum_{p=0}^{\infty}\sum_{q=-p}^{p}\gamma_{pq}(k)Y_{pq}(\hat{\boldsymbol{\phi}}), \tag{39}$$

where $\gamma_{pq}(k)$ is a set of coefficients that completely describe this function.

By substituting (38)and (39) into (37), and also applying the orthogonality property of spherical harmonics [1] and appropriate truncation [33], we get

$$S^{\mathrm{a}}(\boldsymbol{x},k)=\sum_{n=0}^{N}\sum_{m=-n}^{n}ik\gamma_{nm}h_n^{(1)}(kR)j_n(k\boldsymbol{x})Y_{nm}(\hat{\boldsymbol{x}}). \tag{40}$$

To design the loudspeaker aperture function, we equate (40) with the desired soundfield (36) and obtain the relationship between the desired soundfield coefficients and the coefficients of the continuous loudspeaker aperture function

$$\gamma_{nm}(k)=\frac{-i\alpha_{nm}^{(\mathrm{d})}(k)}{kh_n^{(1)}(kR)}. \tag{41}$$

The aperture function of the continuous loudspeaker will now be sampled by a discrete loudspeaker array to permit practical implementation. We place the loudspeakers on the surface of a sphere and all loudspeakers are equidistant from the origin with radius R. Followed the rule of thumb proposed in [1], we observe that if we have $Q\geq(N+1)^2$ points (loudspeakers) on the sphere, and all loudspeakers are equidistant from the origin, where the position of the loudspeakers are given by the library of $3D$ packings at [39], then we can approximately reproduce $\rho(\hat{\boldsymbol{\phi}};k)$ from its samples $\rho_q(\hat{\boldsymbol{\phi}};k)$, $q=1,\ldots,Q$, with some errors.

For accurate 3D sampling, the sampled loudspeaker weight $\rho_q(\hat{\boldsymbol{\phi}};k)$ needs to be scaled by the surface area that they are acting over, denoted as $\triangle SA_{\text{cap}}$. The surface area is calculated by considering the surface area of the cap of a sphere [40]

$$\triangle SA_{\text{cap}}=2\pi rh, \tag{42}$$

and where h is the height calculated from the minimum angle $\triangle\hat{\boldsymbol{\phi}}$ between the adjacent points in the sphere as[13]

$$h=r-\cos(\frac{\triangle\hat{\boldsymbol{\phi}}}{2})\times r. \tag{43}$$

Thus the weight at the Q^{th} loudspeaker to reconstruct the 3D soundfield is:

$$\rho_q(\hat{\boldsymbol{\phi}};k)=\sum_{n=0}^{N}\sum_{m=-n}^{n}\frac{-i\alpha_{nm}^{(\mathrm{d})}(k)}{kh_n^{(1)}(kR)}Y_{nm}(\hat{\boldsymbol{\phi}})\triangle SA_{\text{cap}}. \tag{44}$$

[13]Note this calculation is only an approximation as $\triangle\hat{\boldsymbol{\phi}}$ is not the same for all loudspeakers. This implies that the area each loudspeaker acting over is not symmetrical.

We will not consider the error analysis of 3D case in this paper. Readers may refer to Excell [41] for details.

VI. Simulation

In the following examples, we illustrate the $2D$ height-invariant soundfield reproduction. For the convenience of illustration purposes, we use a planewave source. The examples include reproduction (i) using continuous loudspeaker method, and (ii) conventional least squares method (VI-A). We also illustrate soundfield reproduction due to a cylindrical wave source (VI-B).

A. *Soundfield Reproduction of a planewave using two different methods*

We consider a circular reproduction region of radius 0.5m. The desired soundfield is monochromatic plane wave of frequency of 400 Hz arriving from $45°$. This is equivalent to $kr = 3.696$.

The rule of thumb proposed in [1] suggests $Q \geq 2M + 1$, where $M = \lceil kr \rceil$. This agrees with the sampling theorem to discretize the continuous aperture $Q > 2M$. In these simulations, we are using $M = \lceil kr \rceil$ instead of $M = \lceil erk/2 \rceil$ since we want to verify the consistency on the performance bounds proposed by ward [1] and those of the continuous loudspeaker method. Hence, we choose $M = 4$, thus $Q = 9$ loudspeakers are required. The loudspeakers are equally spaced on a circle of $R = 1$m. The loudspeaker weights are calculated from (20), and the resulting reproduced field using continuous loudspeaker method is shown in Figure 2. It is calculated at 101×101 points and displayed as a "density plot" which means that the numerical values are represented by different shades of gray. Values greater than 1 are white, values less than -1 are black, and values between -1 and 1 are appropriately shaded. The top two plots show the real and imaginary parts of the desired planewave field (without mode limiting), and the bottom two plots show the field reproduced by the loudspeaker array assuming a desired soundfield with mode limiting to M. Figure 3 represents the $2D$ soundfield reproduction using the least squares method for comparison. The reproduced field in Figure 2 corresponds well to the desired field where the boundary of the reproduction area is indicated in the circle. It has the same performance as the soundfield reproduced by the least squares method. The reproduced error in this case is 0.0149 which agrees very well with the expected error of 0.04 referring to [1].

B. *Soundfield reproduction of a cylindrical wave source using continuous loudspeaker method*

In this section, we use different $\alpha_m^{(\mathrm{d})}(k)$ for the cylindrical wave source and keep the same simulation parameters used in VI-A. We obtain the loudspeaker weights and simulate the reproduction of a cylindrical

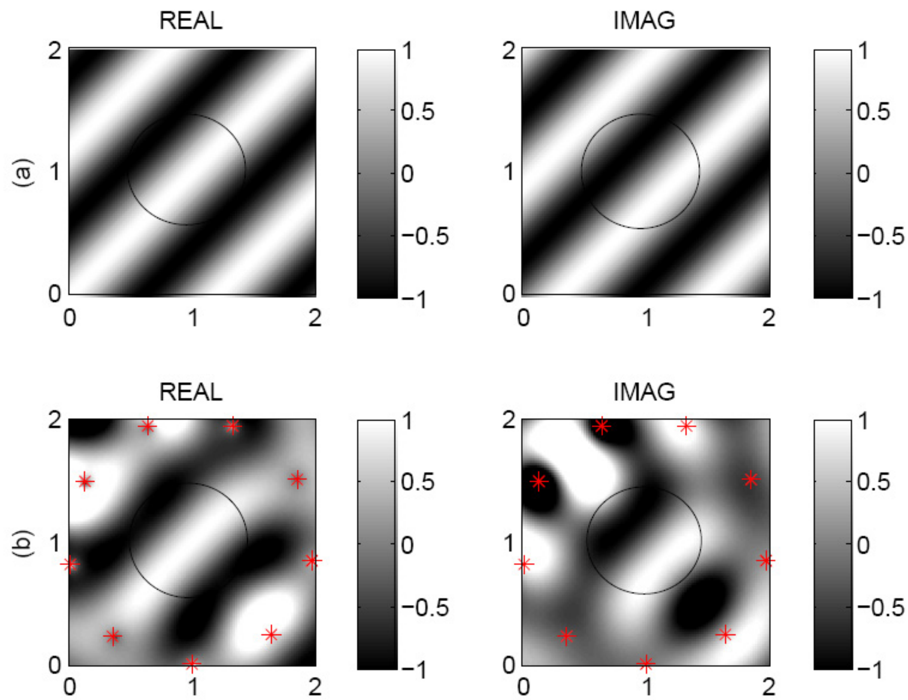

Fig. 2: Reproduction of a 2D plane-wave using the continuous loudspeaker method with 9 loudspeakers for reproduction region of radius 0.5m and frequency of 400Hz, (a) desired field, and (b) reproduction field. The loudspeakers are equally spaced on a circle of $R = 1$m where "$\star$" marks indicate the positions of the loudspeakers.

wave source outside the region of interest, at $\boldsymbol{y} = (1.414, -4/\pi)$. Figure 4 shows a good reproduction of this cylindrical wave source. The reproduction error is 0.0389.

C. Reproduction Error

We define the reproduction error as the average squared difference between the entire desired field $S^{\mathrm{d}}(\boldsymbol{x}, k)$ and the entire corresponding reproduced field $S^{\mathrm{a}}(\boldsymbol{x}, k)$ over the desired reproduction area:

$$\epsilon_M(k, r) = \frac{\int_0^r \int_0^{2\pi} |S^{\mathrm{d}}(\boldsymbol{x}, k) - S^{\mathrm{a}}(\boldsymbol{x}, k)|^2 d\phi_x x dx}{\int_0^r \int_0^{2\pi} |S^{\mathrm{d}}(\boldsymbol{x}, k)|^2 d\phi_x x dx}. \tag{45}$$

Figure 5 shows the reproduction error for different number of loudspeaker Q as a function of kr from the simulation. We make the following comments regarding the result:

A. The reproduced error depends on the product of the wavenumber, k, and the radius of the reproduction area, r. Thus, for a given order M, the higher operating frequency will result in a smaller

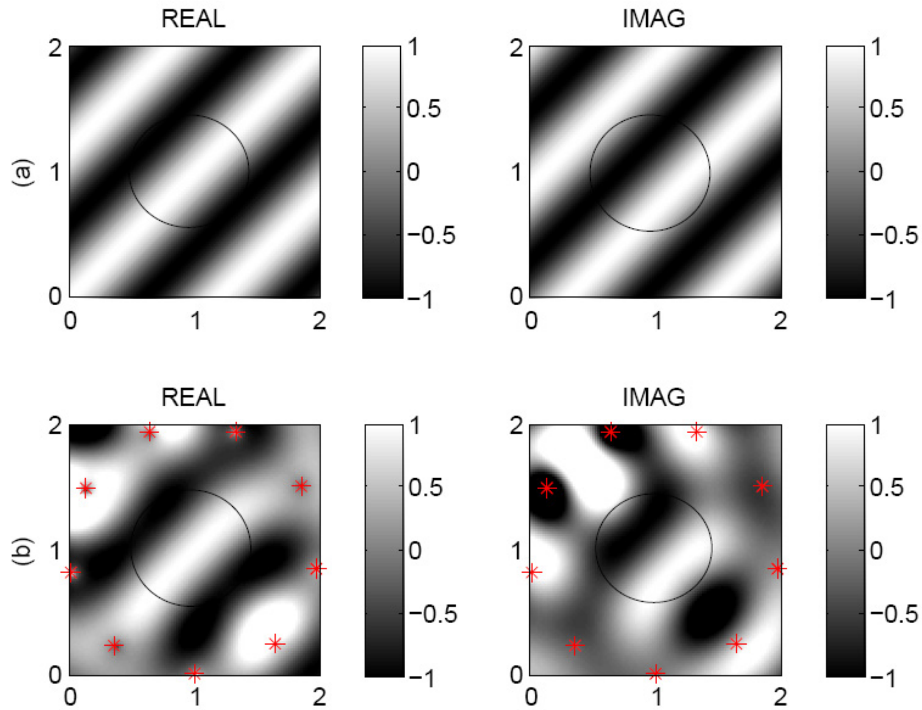

Fig. 3: Reproduction of a 2D plane-wave using the least squares method with 9 loudspeakers for reproduction region of radius 0.5m and frequency of 400Hz, (a) desired field, and (b) reproduction field. The loudspeakers are equally spaced on a circle of $R = 1$m where "$\star$" marks indicate the positions of the loudspeakers.

reproduction area.

B. The reproduced error depends on the number of loudspeakers used and the space between the adjacent loudspeakers. However, we notice when $Q > 2\lceil kr \rceil$, the reproduced error is less than 0.04. Once Q is beyond this threshold, a larger number of loudspeakers only marginally increases the reproduction accuracy. This again validates the rule of thumb proposed in [33] for choosing the required expansion order.

VII. Conclusion

In this paper, we propose a theoretical continuous loudspeaker approach on a circular array to calculate the loudspeaker aperture/weights. We reveal the underlying structure of the loudspeaker aperture/weights as a function of the desired field, the loudspeaker positions and the frequency. We have shown that for a wavenumber k, we can accurately reproduce the sound field within radius r by using a discrete number

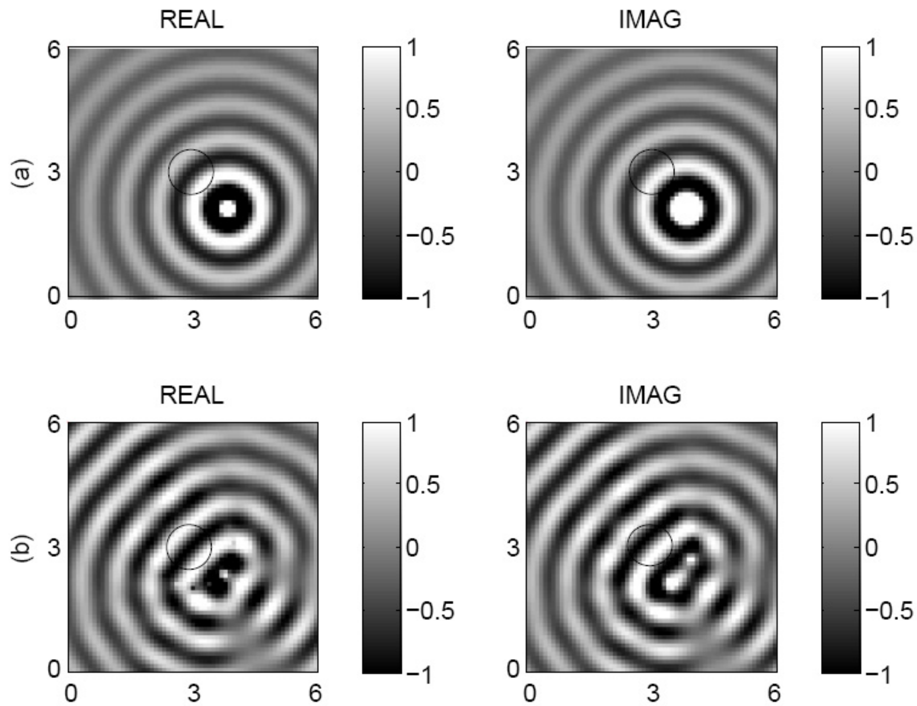

Fig. 4: Reproduction of a 2D cylindrical wave using continuous loudspeaker method with 9 loudspeakers for reproduction region of radius 0.5m and frequency of 400Hz, (a) desired field, and (b) reproduction field. The loudspeakers are equally spaced on a circle of $R = 1$m.

of $Q > 2M + 1$ loudspeakers, where $M = \lceil ker/2 \rceil$. This continuous loudspeaker approach can also be applied to the $3D$ soundfield reproduction using spherical harmonics analysis on a spherical array.

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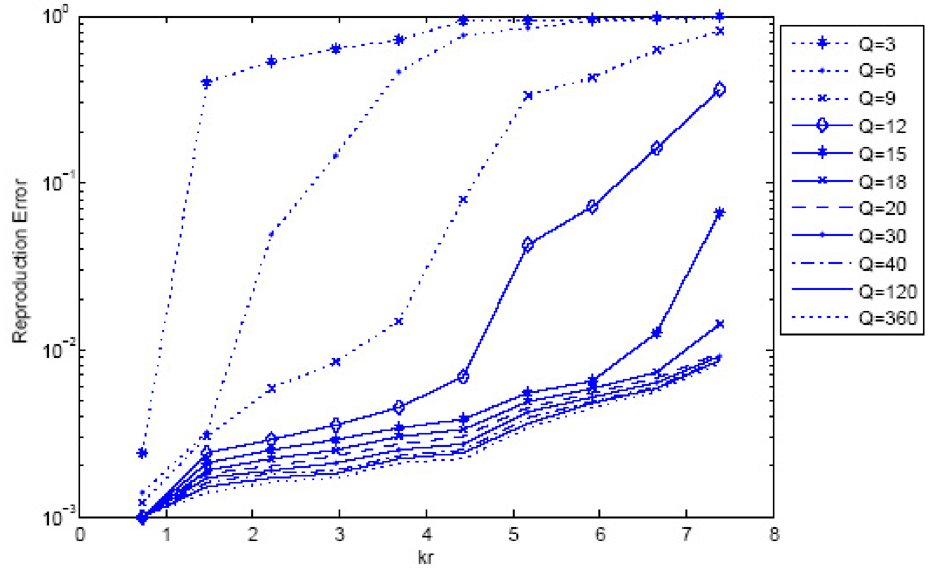

Fig. 5: Reproduction error for a 2D plane-wave field as function of kr for different number of loudspeakers Q using continuous loudspeaker method. The plane-wave frequency is 400Hz. The loudspeakers are equally spaced on a circle of $R = 1$m.

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