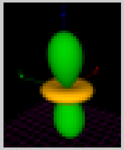
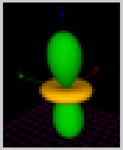
New Dimensions for Ambisonics

Michael Chapman

Paper presented to *AES 124th Convention*
Tuesday May 20th, 2008 — Amsterdam

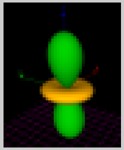


1. Historical Introduction
2. Hyperspherical Harmonics
3. Practicalities
4. Conclusion



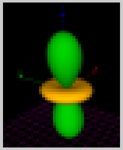
1. Historical Introduction

1783 Legendre —circle

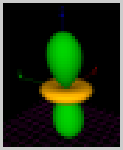


Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Page 3 of 45
Go Back
Print
Close
Quit

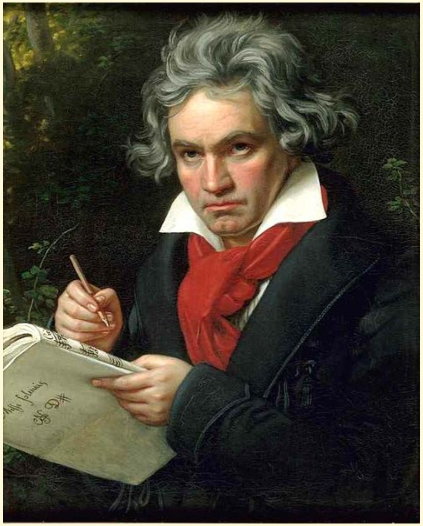
1783 Legendre —circle
1784 Laplace —sphere

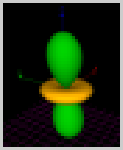


1783 Legendre —circle
1784 Laplace —sphere
1789 French Revolution

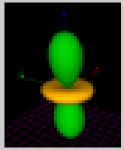


1783 Legendre —circle
1784 Laplace —sphere
1789 French Revolution
1800 first performance of Symphony No. 1, in C, opus 21 of Beethoven



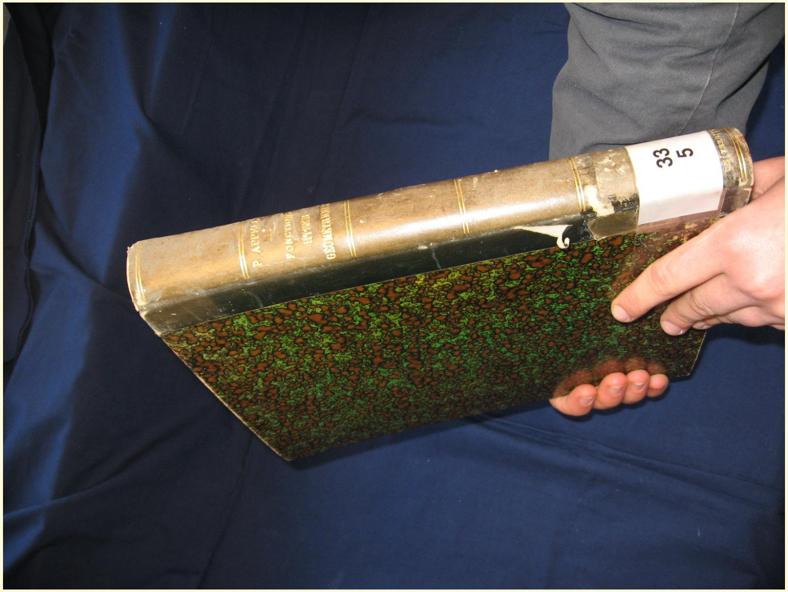


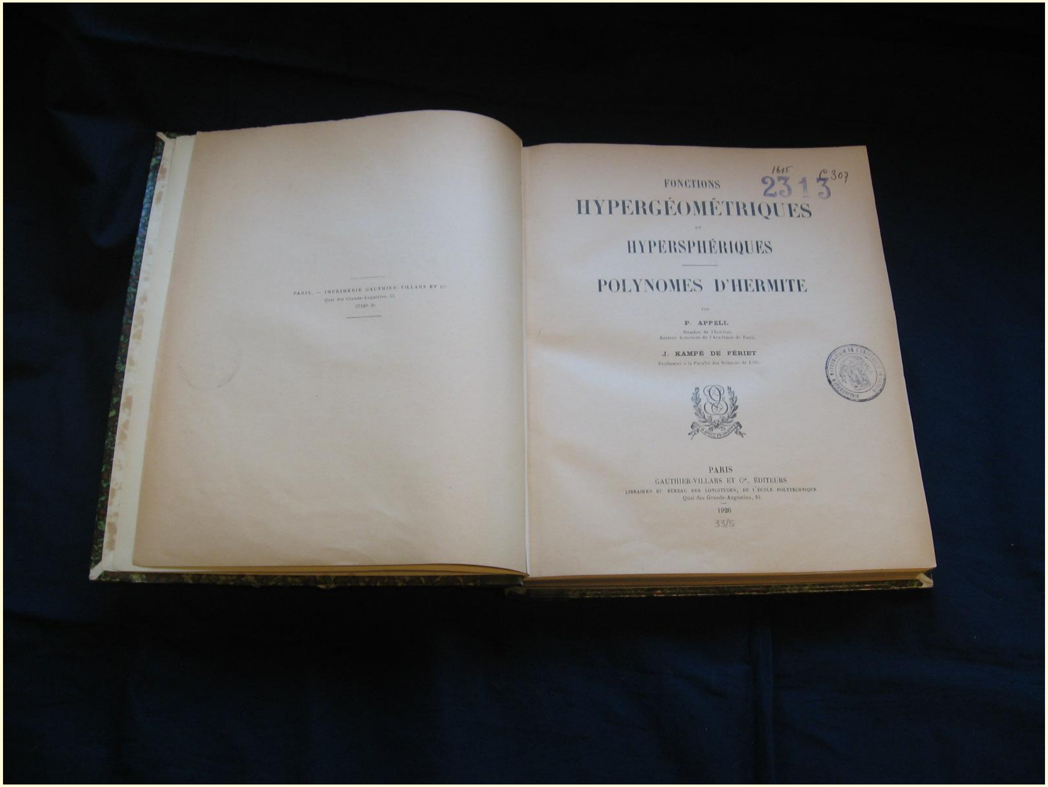
1783 Legendre —circle
1784 Laplace —sphere
1877–1893 Gegenbauer's major publications



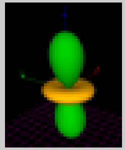
Home Page
Title Page
Page 7 of 45
Go Back
Print
Close
Quit

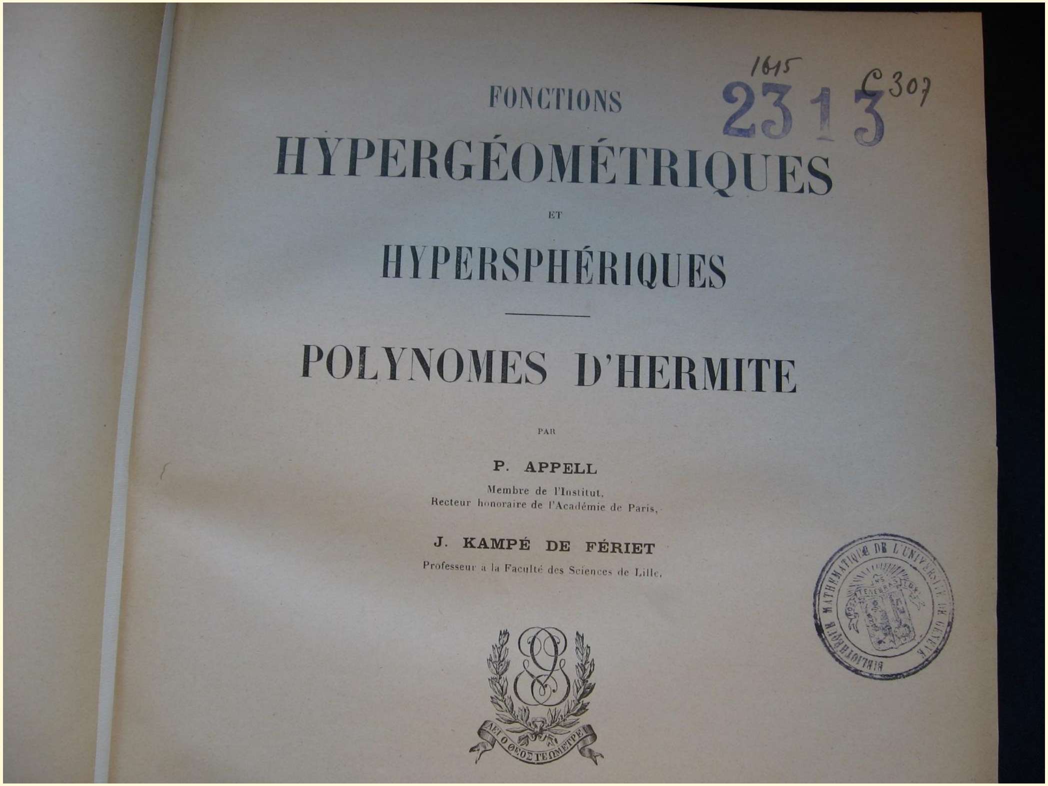
1783 Legendre —circle
1784 Laplace —sphere
1877–1893 Gegenbauer’s major publications
1926 Appell & Kampé de Fériet





Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Page 9 of 45
Go Back
Print
Close
Quit





FONCTIONS

HYPERGÉOMÉTRIQUES

ET

HYPERSPHÉRIQUES

POLYNOMES D'HERMITE

PAR

P. APPELL
Membre de l'Institut,
Recteur honoraire de l'Académie de Paris,

J. KAMPÉ DE FÉRIET
Professeur à la Faculté des Sciences de Lille,

1783 Legendre —circle
1784 Laplace —sphere
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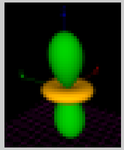

Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes

Home Page
Title Page
Page 11 of 45
Go Back
Print
Close
Quit

1783 Legendre —circle
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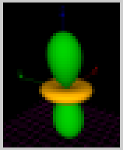
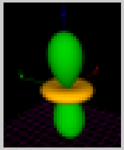
1877–1893 Gegenbauer's major publications
1926 Appell & Kampé de Fériet
1989 & 1999 Avery's books



1783 Legendre —circle
1784 Laplace —sphere
1877–1893 Gegenbauer
1926 Appell & Kampé de Fériet
1989 & 1999 Avery

> “In none of [my] courses was a hypergeometric function mentioned . . . It is small wonder that with a similar education almost all mathematicians think of special functions as a dead subject.”
>
> and
>
> “Like others, I had been put off by all the parameters. . . . so many parameters that it was necessary to put subscripts on them . . . ”.

Richard Askey’s *Foreword* in Gasper and Rahman

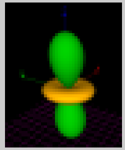


1783 Legendre —circle
1784 Laplace —sphere
1877–1893 Gegenbauer
1926 Appell & Kampé de Fériet

1971
‘Ambisonics’ first used in print

1989 & 1999 Avery

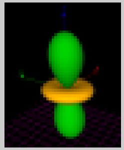
2001
Jérôme Daniel’s thesis



2. Hyperspherical Harmonics

It can be shown that in a d-dimensional universe the number of spherical harmonics of order l, is:

$$(2l+d-2)\frac{(l+d-3)!}{(d-2)!.l!}$$



Home Page
Title Page

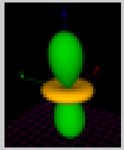
Go Back
Print
Close
Quit

It can be shown that in a d-dimensional universe the number of spherical harmonics of order l, is:

$$(2l + d - 2)\frac{(l + d - 3)!}{(d - 2)!.l!}$$

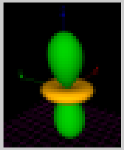
$$d = 2 : \qquad 2$$

$$d = 3 : \qquad 2l + 1$$



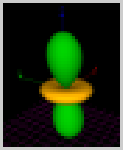
Home Page
Title Page
Page 16 of 45
Go Back
Print
Close
Quit

order	dimensions							
	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
2	2	5	9	14	20	27	35	44
3	2	7	16	30	50	77	112	156
4	2	9	25	55	105	182	294	450
5	2	11	36	91	196	378	672	1,122
6	2	13	49	140	336	714	1,386	2,508
7	2	15	64	204	540	1,254	2,640	5,148
8	2	17	81	285	825	2,079	4,719	9,867
9	2	19	100	385	1,210	3,289	8,008	17,875
10	2	21	121	506	1,716	5,005	13,013	30,888
11	2	23	144	650	2,366	7,371	20,384	51,272
12	2	25	169	819	3,185	10,556	30,940	82,212
13	2	27	196	1,015	4,200	14,756	45,696	127,908
14	2	29	225	1,240	5,440	20,196	65,892	193,800
15	2	31	256	1,496	6,936	27,132	93,024	286,824
16	2	33	289	1,785	8,721	35,853	128,877	415,701



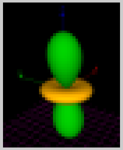
First Order SH

X	$\cos\theta$
Y	$\sin\theta$



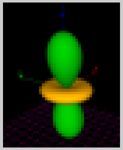
First Order SH

X	$\cos\theta.\cos\phi$
Y	$\sin\theta.\cos\phi$
Z	$\sin\phi$



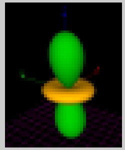
First Order SH

U_1	X	$\cos\theta . \cos\phi$
U_2	Y	$\sin\theta . \cos\phi$
U_3	Z	$\sin\phi$



First Order SH

U_1	X	$\cos\theta . \cos\phi_1 . \cos\phi_2$
U_2	Y	$\sin\theta . \cos\phi_1 . \cos\phi_2$
U_3	Z	$\sin\phi_1 . \cos\phi_2$
U_4		$\sin\phi_2$

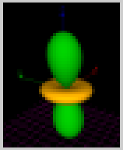


Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page

Go Back
Print
Close
Quit

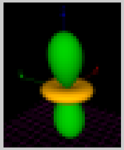
First Order SH

U_1	X	$\cos\theta.\cos\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4$
U_2	Y	$\sin\theta.\cos\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4$
U_3	Z	$\sin\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4$
U_4		$\sin\phi_2.\cos\phi_3.\cos\phi_4$
U_5		$\sin\phi_3.\cos\phi_4$
U_6		$\sin\phi_4$



First Order SH

$$
\begin{aligned}
Y_1^{1,1,1,1} &= \cos\theta.\cos\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,1,1,-1} &= \sin\theta.\cos\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,1,1,0} &= \sin\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,1,0,0} &= \sin\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,0,0,0} &= \sin\phi_3.\cos\phi_4\\
Y_1^{0,0,0,0} &= \sin\phi_4
\end{aligned}
$$



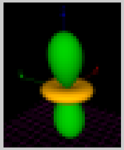
First Order SH

$$
\begin{aligned}
Y_1^{1,1,1,1} &= \cos\theta.\cos\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,1,1,-1} &= \sin\theta.\cos\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,1,1,0} &= \sin\phi_1.\cos\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,1,0,0} &= \sin\phi_2.\cos\phi_3.\cos\phi_4\\
Y_1^{1,0,0,0} &= \sin\phi_3.\cos\phi_4\\
Y_1^{0,0,0,0} &= \sin\phi_4
\end{aligned}
$$

m = range

$$Y_l^m$$

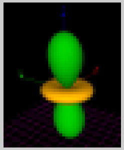
l = order



Second Order SH

U $\quad Y_2^2 = \frac{\sqrt{3}}{2}\cos(2\theta)$

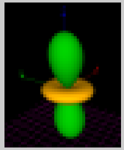
V $\quad Y_2^{-2} = \frac{\sqrt{3}}{2}\sin(2\theta)$



Home Page
Title Page
Page 25 of 45
Go Back
Print
Close
Quit

Second Order SH

$$
\begin{array}{llcl}
\text{U} & Y_2^2 & = & \frac{\sqrt{3}}{2}\cos(2\theta).\cos^2\phi \\
\text{V} & Y_2^{-2} & = & \frac{\sqrt{3}}{2}\sin(2\theta).\cos^2\phi \\
\text{S} & Y_2^1 & = & \frac{\sqrt{3}}{2}\cos(\theta).\sin(2\phi) \\
\text{T} & Y_2^{-1} & = & \frac{\sqrt{3}}{2}\sin(\theta).\sin(2\phi) \\
\text{R} & Y_2^0 & = & \frac{1}{2}(3\sin^2\phi - 1)
\end{array}
$$

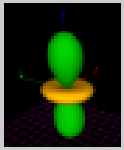


Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes

Home Page
Title Page
Page 26 of 45
Go Back
Print
Close
Quit

Second Order SH

$$
\begin{array}{lrcl}
\mathrm{U} & Y_2^{2,2} & = & \frac{\sqrt{3}}{2}\cos(2\theta).\cos^2\phi_1.\cos^2\phi_2 \\
\mathrm{V} & Y_2^{2,-2} & = & \frac{\sqrt{3}}{2}\sin(2\theta).\cos^2\phi_1.\cos^2\phi_2 \\
\mathrm{S} & Y_2^{2,1} & = & \frac{\sqrt{3}}{2}\cos(\theta).\sin(2\phi_1).\cos^2\phi_2 \\
\mathrm{T} & Y_2^{2,-1} & = & \frac{\sqrt{3}}{2}\sin(\theta).\sin(2\phi_1).\cos^2\phi_2 \\
\mathrm{R} & Y_2^{2,0} & = & \frac{1}{2}(3\sin^2\phi_1 - 1).\cos^2\phi_2 \\
 & Y_2^{1,1} & = & \sqrt{\frac{3}{2}}\cos\theta.\cos\phi_1.\sin(2\phi_2) \\
 & Y_2^{1,-1} & = & \sqrt{\frac{3}{2}}\sin\theta.\cos\phi_1.\sin(2\phi_2) \\
 & Y_2^{1,0} & = & \sqrt{\frac{3}{2}}\sin\phi_1.\sin(2\phi_2) \\
 & Y_2^{0,0} & = & \frac{1}{2.\sqrt{3}}(4\sin^2\phi_2 - 1)
\end{array}
$$



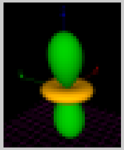
3. Practicalities

3.1. Developments since the preprint

3.1.1. Channel Order

These slides have used Oliver Thuns’ proposal to The Ambisonics Association, for a standardised channel order, basically:

W		
X	U	
Y	V	P
Z	S	Q
	T	N
	R	O
		L
		M
		K

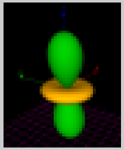


Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Page 28 of 45
Go Back
Print
Close
Quit

3.1.2. Normalisation

As multi-dimesnional files are never going to be played as such, there is no purpose in SN4D (or N4D), SN5D (or N5D), . . .

In these slides SN3D is used.

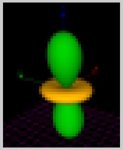


3.2. Rotation

Rotation about one axis is no different from the normal case:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 & \dots & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

can obviously take place around any axis and a rotation about one axis may be applied after a rotation about another axis, to give any position required.



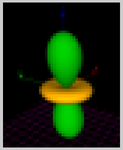
Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 30 of 45
Go Back
Print
Close
Quit

3.3. Gerzon's Dominance

The transformation matrix can be applied to any of the dimensions:

$$\begin{pmatrix} \frac{\lambda+\lambda^{-1}}{2} & \frac{\lambda-\lambda^{-1}}{2} & 0 & \dots & 0 \\ \frac{\lambda-\lambda^{-1}}{2} & \frac{\lambda+\lambda^{-1}}{2} & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$



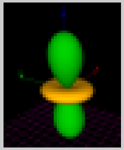
3.4. Selection of a space

Once a hyper-dimensional soundfield is arranged (rotated, or whatever) to suit the listener, it must then be rendered on a real reproduction system. The simplest way of doing this is to ‘drop’ the unwanted channels, that is:

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \end{pmatrix} \times \begin{pmatrix} W \\ U_1 \\ U_2 \\ U_3 \\ U_4 \\ \vdots \end{pmatrix}$$

This approach has been extensively discussed as it is a regular occurrence when periphonic recordings are reproduced on pantophonic rigs. Daniel at page 154 states[1]:

[1] translations by this author, emphasis as in original

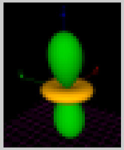


Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Page 32 of 45
Go Back
Print
Close
Quit

> ...the 2D representation is equivalent to a *horizontal "slice" of the 3D representation.* It could be shown that in a 2D restitution, an impression of height can be produced, with lack of up–down resolution. In fact, the modification of the amplitude ratio between the W and X and Y components affects the apparent velocity of the wave front reproduced horizontally (and characterized by the vector velocity $\vec{V}$), and results in the effect of convincing lateralisation for a source outside the horizontal plane (...). A similar interpretation can be applied to the energy vector $\vec{E}$.

later (pages 163-164) he discusses the issue of energy equalisation in relation to mixed dimensional decoding, in greater depth.

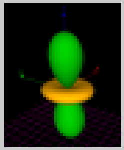
Whether any effect similar to that reported by Daniel, above, can be experienced in periphonic reproduction of higher dimensional material awaits testing on better reproduction systems than the author's.



4. Conclusion

It is not suggested that ambisonics should be market(-ing) led. But even if marketing does not drive, then human ingenuity and human curiosity will seek to discover what can be developed from existing reproduction systems. The owner of an eight-speaker periphonic rig can upgrade the hardware and progress to a second order system with say 12 speakers (or third order with say 32) and all the problems of sub-floor sound sources, or he can use his first order rig in different ways.

Admittedly mixed-order (e.g. ‘fh’ (with channels WXYZUV)) avoids the problem of speakers that are ‘too’ high (or low, for installation). Also Daniel (pages 192–200) discusses possible domed arrays.



Home Page
Title Page

Go Back
Print
Close
Quit

Set against that, first-order hyperambisonics offers:

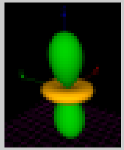
The ability to 'zoom' Gerzon's dominance only works on first-order material.[2]

A modest increase in file sizes A four-dimensional file is 25% larger than a periphonic file.

No need to upgrade playback hardware

In synopsis, the extra 'costs' are in production, rather than in rendering of the work.

[2] see Cotterell (pages 138–144) and, also, the Annex to this paper.

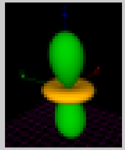


Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Go Back
Print
Close
Quit

Dominance and rotation in hyperspace create possibilities for ‘user interactivity’ with the composition.
Whether that is a ‘good thing’ must depend on the artistic intent of the composer, nevertheless ‘interactivity’ does seem a current trend in many fields.

The techniques described here are only applicable to synthesised material —that is, material specifically composed for ambisonics. They have no relevance to recording real soundfields (though such recorded material may form a part of a synthesised whole).

Such synthesised material does already exist in three-dimensional creations. Jan Jacob Hofmann’s *Sonic Architecture* pieces being a notable example.



5. Annexes

Proof of the uniqueness of rotation

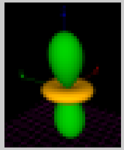
The origin of the method here is acknowledged in the next proof. If we take an generalised pantophonic transformation matrix, such that:

$$\begin{pmatrix} W' \\ X' \\ Y' \end{pmatrix} = \begin{pmatrix} w_w & w_x & w_y \\ x_w & x_x & x_y \\ y_w & y_x & y_y \end{pmatrix} \times \begin{pmatrix} W \\ X \\ Y \end{pmatrix}$$

then for a case where there is no distortion of the soundfield (that is W is unchanged and is not used in forming X' and Y'), we have:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x_x & x_y \\ 0 & y_x & y_y \end{pmatrix}$$

As $W' = W$ (in this case) and as the model must work for a point source where $W^2 = X^2 + Y^2$ and $W'^2 = X'^2 + Y'^2$, then



Home Page
Title Page
Page 37 of 45
Go Back
Print
Close
Quit

as from our transformation matrix we have:

$$X' = x_x X + x_y Y$$

$$Y' = y_x X + y_y Y$$

then we may write

$$X^2 + Y^2 = (x_x X + x_y Y)^2 + (y_x X + y_y Y)^2$$

which may be re-arranged:

$$(x_x^2 + y_x^2 - 1)X^2 + (x_y^2 + y_y^2 - 1)Y^2 + 2(x_x x_y + y_x y_y)XY = 0$$

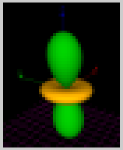
For the above to be true for all values of X and Y the three terms in parentheses must each equal zero. That is:

$$x_x^2 + y_x^2 = 1$$

$$x_y^2 + y_y^2 = 1$$

with, either

$$x_y = y_x \text{ and } x_x = -y_y$$



or:

$$x_y = -y_x \text{ and } x_x = y_y$$

We could write $x_x = \pm\sqrt{1 - y_x^2}$, but we can equally reduce our matrix to having only one variable by substituting $y_x = \sin(\alpha)$ and $y_y = \cos(\alpha)$ (as $\sin^2(\alpha) + \cos^2(\alpha) = 1$ for all α).

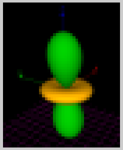
The transformation matrix can now be re-written:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

which is the classic ambisonic rotation matrix. (The alternative solution

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos(\alpha) & \sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

is a rotation but with the soundfield mirrored (or if $\alpha = 0$ just mirroring).).

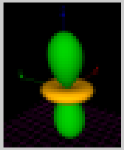


Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Page 39 of 45
Go Back
Print
Close
Quit

Thus the only effects possible with a transformation matrix, as specified above, are rotation and/or mirroring.

It is intuitively obvious that the only non-distorting transformations are rotations and mirrorings, but the above proof does serve as a prelude to the next.

(The above proof can be extended to any number of dimensions: introducing $Z' = Z$, etc., etc. would make no change.)



Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Page 40 of 45
Go Back
Print
Close
Quit

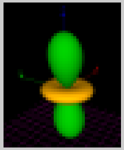
Proof of the uniqueness of Gerzon's dominance

Gerzon & Barton (page 5) establish, for a point source, what we here[3] write as $W^2 = X^2 + Y^2$ and comment besides rotations "there are other linear transformations of a highly non-trivial nature that also satisfy" $W'^2 = X'^2 + Y'^2$ "after transformation. … technically known as *Lorentz transformations*". They then leave it "as an exercise in elementary algebra for the reader to verify" that the dominance transformation they give satisfies $W'^2 = X'^2 + Y'^2$.

There appears to be no published attempt to reverse this analysis and say that as the original signals are ambisonic then when $W^2 = X^2 + Y^2$ is true then what transformations are possible that will result in the output signals satisfying $W'^2 = X'^2 + Y'^2$?

If we take the simple case of a one-dimensional distortion of

[3] Gerzon & Barton's paper is written in 'Gerzon-format', (B-format with FuMa weightings) that is with a weighting factor of $1/\sqrt{2}$ applied to W. Values in the present paper are unweighted.



Home Page
Title Page
Page 41 of 45
Go Back
Print
Close
Quit

the soundfield, then we can write this as:

$$\begin{pmatrix} w_w & w_x & 0 \\ x_w & x_x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using a similar approach to that above, we derive:

$$W^2 - X^2 = (w_w W + w_x X)^2 - (x_w W + x_x X)^2$$

which may, again, be re-arranged:

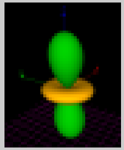
$$(w_w^2 - x_w^2 - 1)W^2 - (x_x^2 - w_x^2 - 1)X^2 + 2(w_w.w_x - x_w.x_x)WX = 0$$

which implies:

$$w_w^2 - x_w^2 = 1$$
$$x_x^2 - w_x^2 = 1$$

with, either

$$w_w = x_x \text{ and } w_x = x_w$$



Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 42 of 45
Go Back
Print
Close
Quit

or:

$$w_w = -x_x \text{ and } w_x = -x_w$$

Again, choosing to use a geometric substitution to reduce the variables to one, as $\csc^2(\alpha) - \cot^2(\alpha) = 1$ for all α, we can write:

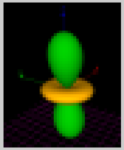
$$\begin{pmatrix} \csc(\alpha) & -\cot(\alpha) & 0 \\ -\cot(\alpha) & \csc(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Defining $t = \tan(\alpha/2)$ then basic geometry gives

$$\csc\alpha = \frac{1+t^2}{2t} = \frac{t^{-1}+t}{2}$$

$$\cot\alpha = \frac{1-t^2}{2t} = \frac{t^{-1}-t}{2}$$

and we can rewrite the matrix as:



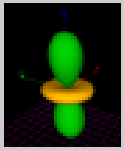
Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes
Home Page
Title Page
Page 43 of 45
Go Back
Print
Close
Quit

$$\begin{pmatrix} \csc\alpha & -\cot\alpha & 0 & 0 \\ -\cot\alpha & \csc\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{t+t^{-1}}{2} & \frac{t-t^{-1}}{2} & 0 & 0 \\ \frac{t-t^{-1}}{2} & \frac{t+t^{-1}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

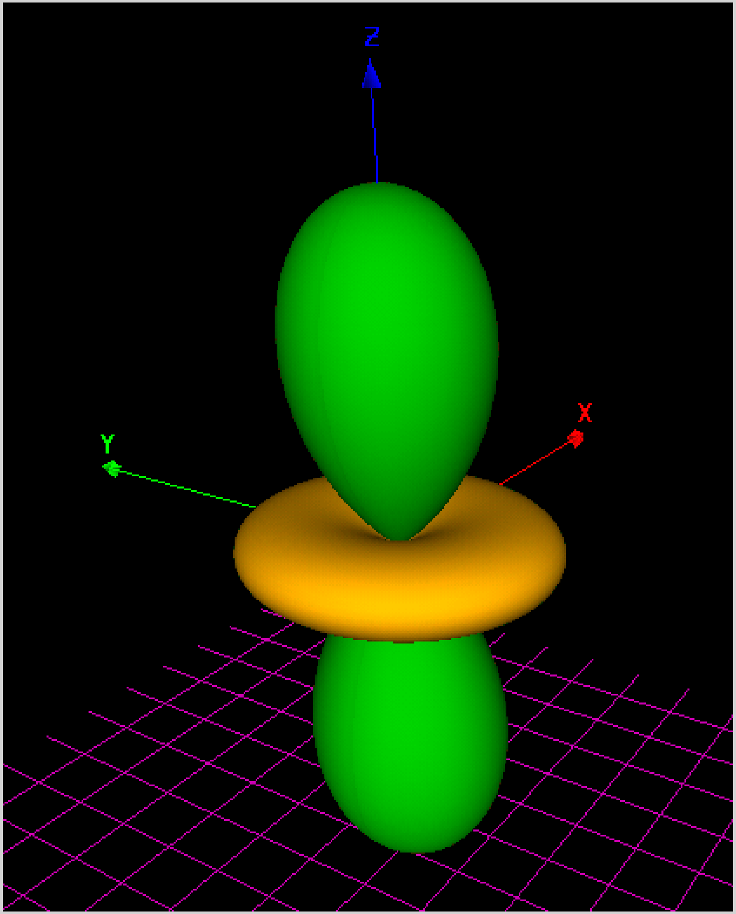
and thus Gerzon's λ is equal to our $\tan(\alpha/2)$.

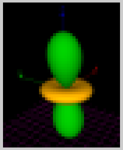
As before we chose only one of the two possible solutions. The other solution gives backwards dominance (or 'undominance' as it is the inverse of the dominance matrix).

(As with the previous proof, extending the number of dimensions does not alter the result.)



Home Page
Title Page
Page 44 of 45
Go Back
Print
Close
Quit





Historical Introduction
Hyperspherical Harmonics
Practicalities
Conclusion
Annexes

Home Page
Title Page
Page 45 of 45
Go Back
Print
Close
Quit