Optimum Loudspeaker Directional Patterns*

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In stereophonic sound reproduction, the loudspeaker placement and directional patterns combine to create the sound fields used by a listener to locate the apparent sources of sound. Accurate localization requires that the loudspeaker directional pattern be designed to take into account the human directional hearing mechanism. A model of human auditory localization is described and then used as a basis for optimizing loudspeaker directional patterns to permit accurate localization over a wide range of listener positions. The results show that loudspeakers should be more directional than current designs, and that the loudspeakers should be angled in toward the listening area.

0 INTRODUCTION

Auditory localization refers to the ability of a listener to determine the direction to a source of sound. In stereophonic sound reproduction the localization behavior of the human auditory system is exploited to give apparent sound sources anywhere in between the two loudspeakers. But the apparent source location perceived by the listener is not always that intended by the record producer or broadcast engineer, and one cause of this problem are the directional patterns of conventional loudspeakers.

The localization performance of a conventional loudspeaker system can be quite poor for a listener positioned off to one side. Consider a center-front soloist or, equivalently, assume that the two loudspeakers are radiating exactly the same waveforms precisely in phase. When the listener is equidistant from the two loudspeakers, the apparent sound source is perceived to be centered between the loudspeakers. This is illustrated in Fig. 1, where O_0 is the centered listener and C_0 is the intended centered image. As the listener moves to the right, the perceived image also moves to the right. This is shown by the succession of images C_1 through C_4, which correspond to listener positions O_0 through O_4. Finally, at listener position O_4, what was intended to be a centered image is perceived as coming from the right loudspeaker instead.

The above example assumes conventional loudspeaker directional patterns, and assumes that the main axes of the loudspeakers are perpendicular to a line drawn from one to the other. Bauer [1], several years ago, realized that localization could be improved by using a loudspeaker having a controlled directional pattern and by angling the loudspeakers in toward the listening area. He proposed a dipole as being a workable solution, and a commercial moving-coil dipole loudspeaker system was designed by Kates *et al.* [2] based on Bauer's ideas. No quantitative testing of the localization properties of the dipole loudspeaker system was conducted, however.

This paper presents a more rigorous solution to the problem of specifying a loudspeaker directional pattern to give good localization performance. We will start by studying human auditory localization, since it is the sound field as perceived by the listener that determines the apparent sound source. We then describe simple mathematical models for the localization for the low-frequency and high-frequency ranges. Using these models, we solve for the loudspeaker directional patterns and loudspeaker orientations that give the best localization performance.

Our conclusions are that loudspeakers should be directional. The ideal spherical source, while being a useful mathematical abstraction, should not be the Holy Grail of loudspeaker design. Instead, loudspeakers should typically have 3-dB bandwidths of from 30 to 90 degrees, and should be angled in toward the listening area so that the loudspeaker axes are aimed at the opposite ends of the desired listening area.

1 AUDITORY LOCALIZATION

Auditory localization is based on comparing the signals received at our two ears. We can model the head as a sphere, with the ears located on opposite sides. The physical separation of the ears means that time differ-

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ences will occur between a signal received at one ear and a signal from the same source received at the other ear. The head is also large enough to cast an acoustic shadow at higher frequencies, and this will cause differences in the intensity of the sound received at one ear in comparison with the other.

In a classic experiment, Stevens and Newman [3] showed that both time differences and intensity differences are used to localize sounds. The time differences are the major cue at low frequencies and intensity differences are the major cue at high frequencies. Mills [4], in a study of the acuity of auditory localization, found that performance was worse in the frequency range of 1500–3000 Hz, and this establishes the boundary between the low-frequency region where temporal cues are the most effective and the high-frequency region where intensity cues are the most effective. Green [5] offers a good review of auditory localization and summarizes much of the available experimental evidence.

1.1 Interaural Time Differences

Interaural time differences can be derived using a simple geometrical model due to Woodworth [6] and shown in Fig. 2. The incoming plane wave reaches one ear unimpeded. The signal to the other ear must travel an additional distance equal to $L_1 + L_2$, where L_1 is the extra distance to the point of tangency and L_2 is the distance along the contour of the head. The geometrical construction gives

$$L_1 = r \sin \phi \tag{1a}$$

$$L_2 = r \phi \ . \tag{1b}$$

The interaural time difference (ITD) is thus

$$\mathrm{ITD} = \frac{r}{c} [\phi + \sin \phi] \tag{2}$$

where

r = radius of head
c = speed of sound in air
ϕ = angle of arrival of plane wave

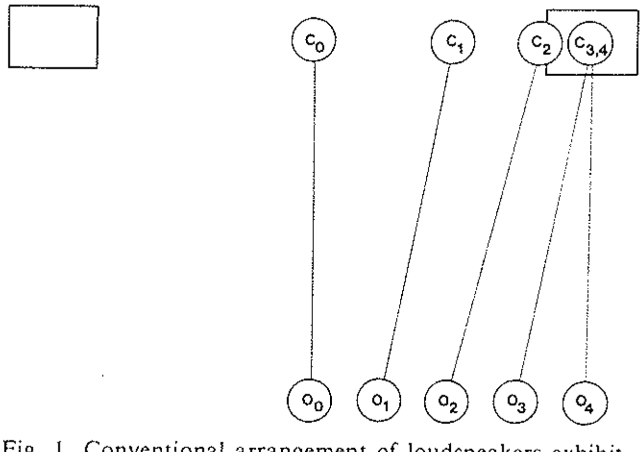

Fig. 1. Conventional arrangement of loudspeakers exhibiting the apparent motion of the central image (after Bauer [1]).

The geometrical solution assumes that the incoming wave follows a specific path around the head. This is true at high frequencies, but Kuhn [7] points out that at lower frequencies one should solve the problem of diffraction of a plane wave around a sphere. His solution to this problem gives

$$\mathrm{ITD} = 3\frac{r}{c} \sin \phi \tag{3}$$

at low frequencies. Since we are primarily interested in the interaural time difference at low frequencies, we will use Eq. (3) in our analysis.

1.2 Interaural Level Differences

Interaural level differences are not as easy to model as interaural time differences. The amount of head shadowing depends on the size of the head relative to the wavelength of the incoming sound wave. At low frequencies the head is relatively small, and very little shadowing takes place. At high frequencies there can be a large amount of shadowing, and additional effects, such as reflections from the pinna (the outer ear), complicate the situation. This is made clear in the data in Fig. 3, taken from the paper by Fedderson *et al.* [8]. At 200 Hz there is essentially no shadowing at all, while at 6000 Hz the interaural intensity difference can be as much as 20 dB. The curves are not very smooth, and there is a pronounced asymmetry about 90 degrees caused by the pinna.

In view of the complexity of the measured results, we

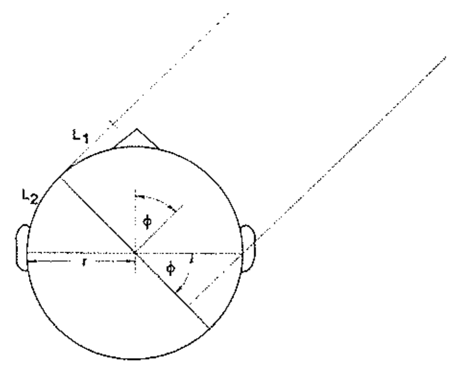

Fig. 2. Path length difference ($L_1 + L_2$) between the two ears for a distant source at an azimuth ϕ (after Woodworth [6]).

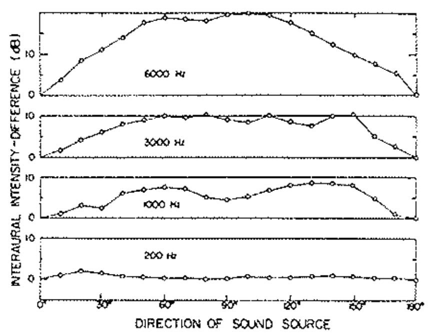

Fig. 3. Interaural intensity difference measured at the two ears as a function of the azimuth of the sound source (after Fedderson *et al.* [8, p. 989]).

would not expect to be able to fit a very accurate mathematical model. Instead, we will use an approximation to the data that shows the general trends while ignoring the fine structure. Bernfeld and Smith [9] approximate the data of Fig. 3 for the interaural level difference (ILD) as

$$\mathrm{ILD} = 1 + f^{0.8} \sin \phi \tag{4}$$

where
f = signal frequency, in kHz
ϕ = angle of arrival of plane wave
We will use Eq. (4) for the interaural level differences.

2 EQUATIONS FOR OPTIMAL DIRECTIONAL PATTERNS

The optimal loudspeaker directional patterns depend on the loudspeaker positions and the range of allowable listener locations. We will assume a single stereo pair of loudspeakers separated by a distance $2D_1$, as shown in Fig. 4. The listener is constrained to a line a distance Y away from the loudspeakers, and the allowable listener locus is $-D_2 \leq x < D_2$. The loudspeakers are at an angle α from the line that connects them. The listener is at a distance r_L from the left loudspeaker, and is located at an angle θ_L from the left loudspeaker axis. Similarly, the listener is at a distance r_R from the right loudspeaker, and is located at an angle θ_R from the right loudspeaker axis.

We will assume that the signal being fed to the loudspeakers represents a center-front acoustic image at all frequencies when the listener is located at $x = 0$, that is, equidistant from the two loudspeakers. This requires that the phase and amplitude responses of the loudspeakers be very closely matched, and we will assume that they are identical for the purposes of this analysis. As the listener moves away from the centered position, he continues to look at the spot halfway between the loudspeakers. This gives an angle of ϕ_L for the left source relative to the listener and an angle of ϕ_R for the right source. A further assumption is that the loudspeakers and listener are in an anechoic environment, so that the loudspeaker directional patterns completely control the localization performance.

Our goal is that no matter where the listener is located along the locus $-D_2 \leq x \leq D_2$, the center-front image will remain centered between the two loudspeakers.

2.1 Low-Frequency Solution

Localization at low frequencies is achieved by comparing the phases of the signals at the two ears. We will assume that the two loudspeakers are exactly in phase and are emitting a pure tone. Since the listener is facing a point halfway between the two loudspeakers, a centered image corresponds to having the signals received at the two ears be exactly in phase.

Let $Q(\theta)$ be the loudspeaker directional pattern. The signal intensity at the listener's location is then $Q(\theta_L)/r_L$ for the left loudspeaker and $Q(\theta_R)/r_R$ for the right loudspeaker. Following through the steps of the deviation in Appendix 1 gives us

$$\frac{Q(\theta_R)}{Q(\theta_L)} = \frac{r_R}{r_L} \cdot \frac{\mathrm{ITD}_L}{\mathrm{ITD}_R} \tag{5}$$

where
ITD_L = interaural level difference due to left source
ITD_R = interaural level difference due to right source
Substituting Eq. (3) into Eq. (5) then gives

$$\frac{Q(\theta_R)}{Q(\theta_L)} = \frac{r_R}{r_L} \cdot \frac{\sin \phi_L}{\sin \phi_R} \tag{6}$$

as the ratio of the directional patterns at the listener's location that will give rise to a centered image.

Our solution of Eq. (6) specifies the ratio of the directional patterns, but not the directional patterns themselves. There are an infinite number of patterns that can satisfy Eq. (6). One solution that gives relatively broad beamwidths, and therefore should be relatively easy to realize with existing technology, is to set

$$Q(\theta_R) = k\, r_R \sin \phi_L \tag{7a}$$

$$Q(\theta_L) = k\, r_L \sin \phi_R \tag{7b}$$

where k is an arbitrary constant.

We can immediately draw some conclusions from Eqs. (7). In most situations the extreme listener location is directly opposite one of the loudspeakers, that is, $D_2 = D_1$. Consider, for example, a listener at $x = -D_2$ who is therefore directly opposite the left loudspeaker. In this situation, both r_R and $\sin \phi_L$ are at their maximum values, while r_L and $\sin\phi_R$ are at their minimum values. Thus $Q(\theta_R)$ is at its maximum and $Q(\theta_L)$ is at its minimum. Given a loudspeaker with a symmetric radiation pattern, the obvious conclusion is that the maximum of the directional pattern, that is, the main axis of the loudspeaker, should be aimed at the far end of the listener locus. The left loudspeaker should be aimed at the point $x = D_2$, and the right loudspeaker should be aimed at the point $x = -D_2$.

2.2 High-Frequency Solution

Localization at high frequencies is achieved by comparing the intensities of the signals at the two ears. We will assume that the two loudspeakers are in phase and are radiating identical narrow-band noise signals. Since the listener is facing a point halfway between the two loudspeakers, a centered image corresponds to having exactly the same received intensity at each ear.

We are using a high-frequency noise source as our test signal. Since we are dealing with path-length differences that are large compared with the wavelengths being radiated, we can assume that the signals from the two loudspeakers add incoherently at each ear. Solving for equal intensities at each ear gives

$$\frac{Q(\theta_R)}{Q(\theta_L)} = \frac{r_R}{r_L} \cdot \left[\frac{1 - (1/\mathrm{ILD}_L)^2}{1 - (1/\mathrm{ILD}_R)^2}\right]^{\frac{1}{2}} \tag{8}$$

where
ILD_L = interaural time difference due to left source

$\mathrm{ILD}_{\mathrm{R}}$ = interaural time difference due to right source

Using the approximation of Eq. (4) leads to the solutions

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}} \cdot \left[\frac{\sin \phi_{\mathrm{L}}}{\sin \phi_{\mathrm{R}}}\right]^{\frac{1}{2}}; \qquad f << 1 \tag{9}$$

and

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}}; \qquad f >> 1 \ . \tag{10}$$

The solution for $f \approx 1$ kHz lies in between those of Eqs. (9) and (10). A derivation is given in Appendix 2.

The high-frequency solution has a very simple form for $f >> 1$. In terms of interaural level differences, the limit of very high frequencies is equivalent to having almost infinite head-shadowing attenuation. Thus the left ear hears only the left loudspeaker, and the right ear hears only the right loudspeaker. Under these circumstances the loudspeaker directional patterns need only compensate for the power loss due to spherical spreading. This gives the directional patterns

$$Q(\theta_{\mathrm{R}}) = k\, r_{\mathrm{R}} \tag{11a}$$

$$Q(\theta_{\mathrm{L}}) = k\, r_{\mathrm{L}} \tag{11b}$$

where again k is an arbitrary constant.

At lower frequencies the solution lies in between the time-difference solution of Eq. (6) and the limiting solution of Eq. (10). In fact, Eq. (9) can be considered to be the geometric mean between the other two solutions since the square root of the $\sin\phi$ terms appears in the solution. This means that directional patterns at intermediate frequencies should lie in between those computed using Eq. (6) for the low frequencies and those computed using Eq. (10) for the highest frequencies.

3 CALCULATION OF OPTIMAL DIRECTIONAL PATTERNS

There is no single optimal loudspeaker directional pattern. The low-frequency and high-frequency solutions depend on the geometry of the loudspeaker positions and the listener locus. Even for a specified geometry we have uniquely determined the ratio of the directional patterns but not the patterns themselves, and this gives an additional amount of freedom in selecting the loudspeaker design.

In this section we present optimal directional pattern calculations for several different geometries. The patterns are based on Eqs. (7) for the low-frequency solution and Eqs. (11) for the high-frequency solution. In Figs. 5–8 the solid line represents the low-frequency directional pattern and the dashed line represents the high-frequency directional pattern; we would expect mid-frequency solutions to lie between these two curves. The directional patterns are specified only for the angular region that irradiates the listener locus, and we have assumed that the radiation patterns are symmetrical about the loudspeaker axis $\theta = 0$. In keeping with the discussion at the end of Section 2.1, the loudspeaker axis is aimed toward the opposite extreme of the listener locus, as shown in Fig. 4.

In Fig. 5 we present the directional patterns for a listener locus far from the loudspeakers. The distance Y to the listener locus is equal to $4D_1$, which would correspond to the listener on a sofa 6 m away from loudspeakers separated by a total distance of 3 m. The total extent of the listener locus is $2D_2$, and this is the same as the loudspeaker separation, that is, 3 m in our example. In this case the loudspeaker directional pattern at low frequencies is down about 2 dB at 27 deg off axis, and the high-frequency pattern is about 1 dB down at the same angle. The high-frequency pattern can be roughly approximated by $\cos\theta$, and the low-frequency pattern has approximately a $\cos^2\theta$ dependence over the angular extent of 0–27 deg that corresponds to the listener locus.

The listener locus is moved closer to the loudspeaker in Fig. 6, where we have $D_2 = D_1 = Y/3$. Both the low-frequency and the high-frequency patterns are more

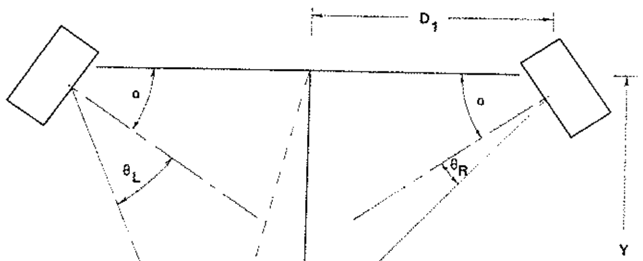

Fig. 4. Listener and loudspeaker geometry used to derive the optimal loudspeaker directional patterns.

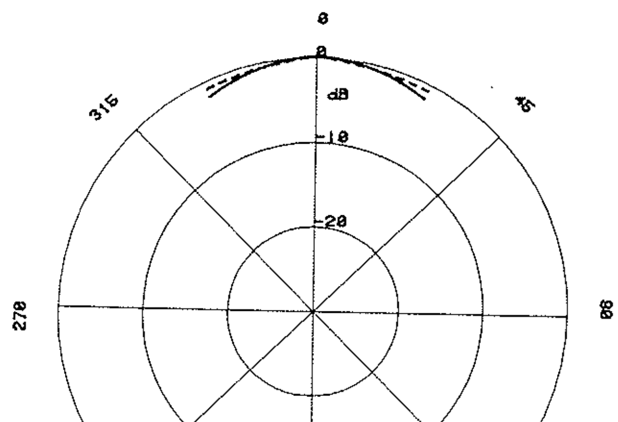

Fig. 5. Optimal directional patterns for $D_2 = D_1 = Y/4$. The loudspeaker orientation is α = 63 deg. Solid line—low-frequency solution; dashed line—high-frequency solution.

directional than those of Fig. 5. The low-frequency pattern is 3 dB down at 32 deg, and is therefore roughly approximated by $\cos^2\theta$. The high-frequency pattern reaches about 1.6 dB down at 32 deg, and the low-frequency and high-frequency curves cross at 15 deg.

The listener locus is again moved closer to the loudspeakers in Fig. 7. Here $D_1 = D_1 = Y/2$, which means that the distance of the listener locus from the loudspeakers is the same as the total loudspeaker separation. The directional pattern is now specified over a +45 to −45 deg region, and the loudspeaker orientation is also 45 deg. The low-frequency pattern is 6 dB down at 45 deg and 3 dB down at 24.5 deg, and the high-frequency pattern is 3 dB down at 45 deg and 2 dB down at 18 deg. This is much more directional than the previous examples.

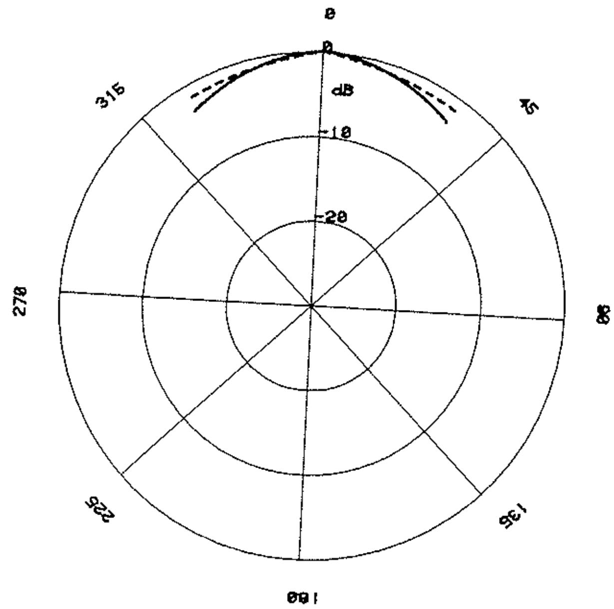

Fig. 6. Optimal directional patterns for $D_2 = D_1 = Y/3$. The loudspeaker orientation is $\alpha = 56$ deg.

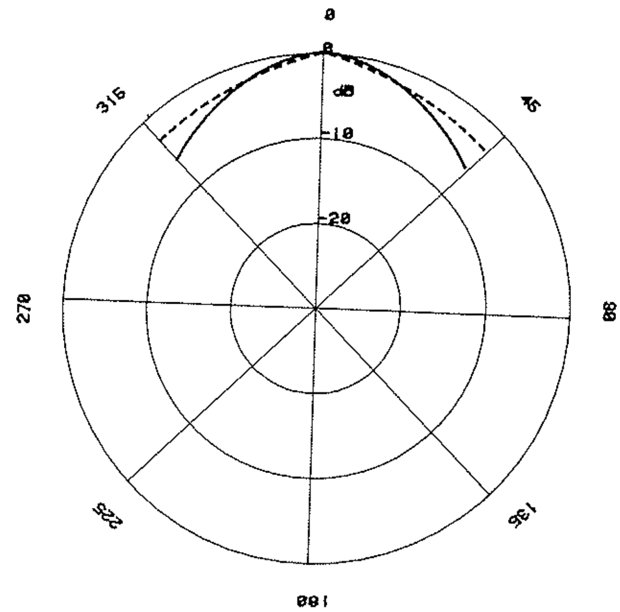

Fig. 7. Optimal directional patterns for $D_2 = D_1 = Y/2$. The loudspeaker orientation is $\alpha = 45$ deg.

Moving the listener locus even closer to the loudspeakers causes even more extreme behavior. In Fig. 8 we have $D_2 = D_1 = Y$. The low-frequency pattern is 3 dB down at 15 deg, 6 dB down at 36 deg, and 14 dB down at 63 deg. The high-frequency pattern crosses the low-frequency pattern at 18 deg and reaches a minimum of 7 dB down at the extreme of 63 deg. Such strong directional behavior is unusual in commercially available loudspeaker systems.

The results of our calculations are summarized in Table 1. The 3-dB beamwidth is defined as the total angular extent from one 3-dB down point to the other on the directional pattern. The obvious trend in the data is that the loudspeakers should become more directional as the listener locus is moved closer to them. In addition, the optimal directional patterns are frequency dependent, with the high-frequency patterns having a sharper point near the loudspeaker axis but not falling off as much at the edges of the directional pattern when compared with the low-frequency patterns.

4 CONCLUSIONS

Loudspeaker design involves both the physics of transducer construction and the psychophysics of human auditory perception. Perceptual models should form the basis of the loudspeaker design goals, since it is the sound field as perceived by the listener that is important.

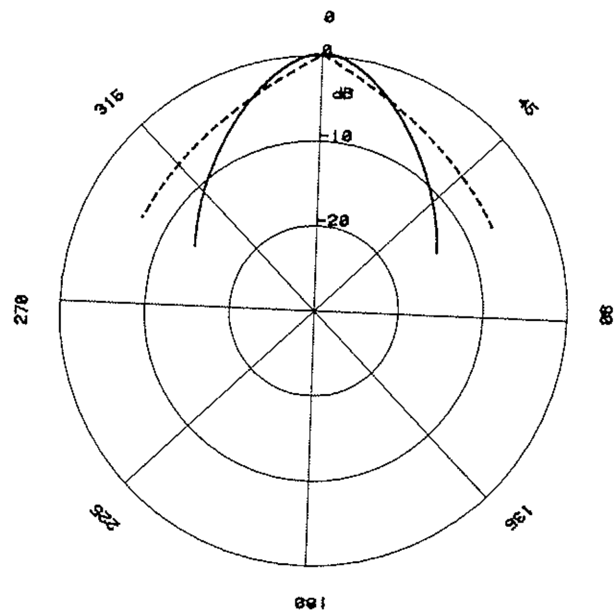

Fig. 8. Optimal directional patterns for $D_2 = D_1 = Y$. The loudspeaker orientation is $\alpha = 27$ deg.

Table 1. Summary of results of optimal directional pattern calculations for $D_2 = D_1$. The approximate beamwidths are extrapolated from Figs. 5 and 6.

D_1/Y	Orientation α, [deg]	3-dB Beamwidth Low-Frequency	High-Frequency
4	63	~ 90	—
3	56	64	~ 90
2	45	49	49
1	27	30	25

In this paper we have used low-frequency and high-frequency models of auditory localization to derive optimal loudspeaker directional patterns.

We have made several simplifying assumptions in our analysis. One is that we dealt only with a center-front image. Noncentered sound sources also occur, and the microphone arrangement used to record the sound can have a strong effect on the perceived location of these acoustic images. Bernfeld and Smith [9] have studied the localization effects of different microphone arrangements for a centered listener, and we would expect additional shifts in the perceived source location due to moving the listener. Using directional loudspeakers will minimize these effects by anchoring the center-front image, but more work is clearly needed in order to optimize the localization performance of the complete sound-reproduction system from microphone to listener.

We have also assumed an anechoic environment for this analysis. Loudspeakers are normally used in rooms, and Queen [10] has shown that reflections from the room surfaces will degrade localization performance. Making the loudspeakers more directional should reduce the energy in the reflections, and should therefore improve this aspect of localization as well. If ambience enhancement using reflections is desired as part of a sound system, a time delay unit should be used in order not to compromise localization of the direct sounds from the loudspeakers. A similar argument would suggest that the rear radiation from a dipole loudspeaker should be absorbed if the best possible localization is the design goal.

The result of our analysis is that loudspeakers should be more directional than conventional design practice. For a listener located at a distance that is 1½ to 2 times the loudspeaker separation, a cosine pattern (front lobe) will give good localization performance. For a listener locus closer to the loudspeakers, the loudspeaker beamwidths must be narrower to give accurate localization over the range of listener positions. Combined with the increase in directionality is a requirement that the loudspeakers be angled in toward the listening area so that the respective maxima of the directional patterns are aimed at the diametrically opposite ends of the listener locus.

An ideal spherical source has traditionally been the goal of many loudspeaker designers. We have shown, however, that a consideration of auditory perception leads to a different design goal. Loudspeakers that are directional over a wide frequency range are required to give the best localization performance.

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APPENDIX 1 DERIVATION OF THE LOW-FREQUENCY SOLUTION

The derivation will use the same symbols that were defined in relation to Fig. 4. We will need some additional symbols in order to describe the distance from each loudspeaker to each ear. The distances are:

r_L = distance from left loudspeaker to center of head
r_{LR} = distance from left loudspeaker to right ear
r_{LL} = distance from left loudspeaker to left ear
r_R = distance from right loudspeaker to center of head
r_{RR} = distance from right loudspeaker to right ear
r_{RL} = distance from right loudspeaker to left ear

The signal being radiated from the loudspeakers can be expressed as a complex sinusoid:

$$s(t) = \exp[j\omega t] \; . \tag{12}$$

The signal received at the left ear due to the two sources is then

$$e_L(t) = \exp\left[j\omega\left(t - \frac{r_{LL}}{c}\right)\right] \cdot \left\{\frac{1}{r_L} Q(\theta_L) + \frac{1}{r_R} Q(\theta_R) \exp\left[-j\omega\left(\frac{r_{RR}}{c} - \frac{r_{LL}}{c} + \mathrm{ITD_R}\right)\right]\right\} \tag{13}$$

and the signal received at the right ear is

$$e_R(t) = \exp\left[j\omega\left(t - \frac{r_{LL}}{c}\right)\right] \cdot \left\{\frac{1}{r_R} Q(\theta_R) \exp\left[-j\omega\left(\frac{r_{RR}}{c} - \frac{r_{LL}}{c}\right) + \frac{1}{r_L} Q(\theta_L) \exp[-j\omega(\mathrm{ITD_L})]\right\} \; . \tag{14}$$

The leading term in both expressions is the same, and can therefore be ignored in the phase comparison. This gives us

$$\angle e_{\mathrm{L}}(t) = \tan^{-1}\left[\frac{\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})\sin\omega\left(\dfrac{r_{\mathrm{RR}}}{c}-\dfrac{r_{\mathrm{LL}}}{c}+\mathrm{ITD_R}\right)}{\dfrac{1}{r_{\mathrm{L}}}Q(\theta_{\mathrm{L}})+\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})\cos\omega\left(\dfrac{r_{\mathrm{RR}}}{c}-\dfrac{r_{\mathrm{LL}}}{c}+\mathrm{ITD_R}\right)}\right] \tag{15}$$

and

$$\angle e_{\mathrm{R}}(t) = \tan^{-1}\left[\frac{\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})\sin\omega\left(\dfrac{r_{\mathrm{RR}}}{c}-\dfrac{r_{\mathrm{LL}}}{c}\right)+\dfrac{1}{r_{\mathrm{L}}}Q(\theta_{\mathrm{L}})\sin\omega\,(\mathrm{ITD_L})}{\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})\cos\omega\left(\dfrac{r_{\mathrm{RR}}}{c}-\dfrac{r_{\mathrm{LL}}}{c}\right)+\dfrac{1}{r_{\mathrm{L}}}Q(\theta_{\mathrm{L}})\cos\omega\,(\mathrm{ITD_L})}\right]. \tag{16}$$

For small time differences and/or low frequencies we can approximate $\sin x \approx x$ and $\cos x \approx 1$. This gives us

$$\angle e_{\mathrm{L}}(t) \approx \frac{\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})\left(\dfrac{r_{\mathrm{RR}}}{c}-\dfrac{r_{\mathrm{LL}}}{c}+\mathrm{ITD_R}\right)}{\dfrac{1}{r_{\mathrm{L}}}Q(\theta_{\mathrm{L}})+\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})} \tag{17}$$

and

$$\angle e_{\mathrm{R}}(t) \approx \frac{\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})\left(\dfrac{r_{\mathrm{RR}}}{c}-\dfrac{r_{\mathrm{LL}}}{c}\right)+\dfrac{1}{r_{\mathrm{L}}}Q(\theta_{\mathrm{L}})\,\mathrm{ITD_L}}{\dfrac{1}{r_{\mathrm{L}}}Q(\theta_{\mathrm{L}})+\dfrac{1}{r_{\mathrm{R}}}Q(\theta_{\mathrm{R}})}. \tag{18}$$

A centered image requires that $\angle e_{\mathrm{L}}(t) = \angle e_{\mathrm{R}}(t)$. Setting Eq. (17) equal to Eq. (18) and canceling the common terms, gives us

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}}\,\frac{\mathrm{ITD_L}}{\mathrm{ITD_R}}. \tag{19}$$

Using Eq. (3) for the interaural time delays gives the solution

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}}\,\frac{\sin\phi_{\mathrm{L}}}{\sin\phi_{\mathrm{R}}}. \tag{20}$$

APPENDIX 2 DERIVATION OF THE HIGH-FREQUENCY SOLUTION

The noise signals received at each ear add incoherently. The average power at the left ear is the sum of the powers from the two sources, taking the head shadowing into account:

$$\langle e_{\mathrm{L}}^2(t)\rangle = \frac{1}{r_{\mathrm{L}}^2}Q^2(\theta_{\mathrm{L}}) + \frac{1}{(\mathrm{ILD_R})^2}\,\frac{1}{r_{\mathrm{R}}^2}Q^2(\theta_{\mathrm{R}}). \tag{21}$$

Similarly we have at the right ear

$$\langle e_{\mathrm{R}}^2(t)\rangle = \frac{1}{r_{\mathrm{R}}^2}Q^2(\theta_{\mathrm{R}}) + \frac{1}{(\mathrm{ILD_L})^2}\,\frac{1}{r_{\mathrm{L}}^2}Q^2(\theta_{\mathrm{L}}). \tag{22}$$

Setting the two received powers equal to give a perceived centered image,

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}}\left[\frac{1-(1/\mathrm{ILD_L})^2}{1-(1/\mathrm{ILD_R})^2}\right]^{\frac{1}{2}}. \tag{23}$$

The interaural level differences can be approximated by Eq. (4), giving

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}}\left[\frac{\sin\phi_{\mathrm{L}}}{\sin\phi_{\mathrm{R}}}\right]^{\frac{1}{2}}\left[\frac{1+f^{0.8}\sin\phi_{\mathrm{R}}}{1+f^{0.8}\sin\phi_{\mathrm{L}}}\right]^{\frac{1}{2}}. \tag{24}$$

At low frequencies the $f^{0.8}$ dependence drops out, so

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}}\left[\frac{\sin\phi_{\mathrm{L}}}{\sin\phi_{\mathrm{R}}}\right]^{\frac{1}{2}} \tag{25}$$

while at high frequencies the sine dependence cancels out, giving

$$\frac{Q(\theta_{\mathrm{R}})}{Q(\theta_{\mathrm{L}})} = \frac{r_{\mathrm{R}}}{r_{\mathrm{L}}}. \tag{26}$$

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